and for each particular $f$ the measure may be so chosen that (moreover)

$$\int_B \log |f(x)| \, m_\nu(dx) \geq \log |f(s)|.$$

Naturally, if for some reason there is for some $s$ only one measure satisfying 7.12, then 7.13 holds for that measure.

**Bibliography**


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**FAMILIES OF CURVES**

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Amasa Forrester in [1] proved the following theorem of a mixed Euclidean and topological character. If $\phi$ is a continuous map without fixed points on the Euclidean $n$-sphere such that $\phi^2$ is the identity, then the chords $P\phi(P)$ for all points $P$ of the sphere completely fill the interior of this sphere.

The object of this note is to generalize this theorem to a purely topological statement.

First we recall the definition of retract. If $B \subseteq A$ are two spaces, then $B$ is a retract of $A$ if there is $r: A \rightarrow B$ which leaves fixed all points of $B$. (If $X$ and $Y$ are spaces the symbol $f: X \rightarrow Y$ shall denote a continuous map from $X$ to $Y$.)

Let $I$ denote the unit interval. If $F: B \times I \rightarrow A$ and $t \in I$, define $F_t: B \rightarrow A$ by $F_t(b) = F(B, t)$ for all $b \in B$.

**Observation.** If $F: B \times I \rightarrow A$ and if $B$ is a retract of $A$ by the map $r$ and if $p, q \in I$, then $rF_p$ is homotopic to $rF_q$.

In fact such a homotopy is provided by $G: B \times I \rightarrow B$ defined by

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\[ G(b, t) = rF(b, p + (q - p)t). \]

Clearly \( G_0 = rF_p \) and \( G_1 = rF_q \).

Now let \( E^{n+1} \) be the topological \( n+1 \) dimensional cell and \( S^n \) its boundary (an \( n \) dimensional topological sphere). Now, for any point \( P \in E^{n+1} - S^n \), \( S^n \) is a retract of \( E^{n+1} - \{ P \} \) by the map

\[ r: E^{n+1} - \{ P \} \to S^n \]

defined by carrying over the central projection of the Euclidean cell by a homeomorphism. This fact and the observation yield:

**Proposition 1.** If \( f_i: S^n \to S^n, i = 0, 1, \) are not homotopic and if \( F: S^n \times I \to E^{n+1} \) satisfies \( F(P, i) = f_i(P), i = 0, 1, \) all \( P \in S^n \), then \( F \) is onto \( E^{n+1} \).

**Proposition 2** (generalization of Forrester’s theorem). Let \( \phi: S^n \to S^n \) be of period \( p \neq 1 \). Let \( F: S^n \times I \to E^{n+1} \) satisfy (a) \( F(P, 0) = P \) and (b) \( F(P, 1) = F(\phi(P), 1) \). Then \( F \) is onto \( E^{n+1} \).

**Proof.** Observe first that it is sufficient to prove this for \( p \) prime. For if \( p \) were not prime and \( q \) is a prime dividing \( p \), then the hypothesis of Proposition 2 is satisfied with \( \phi \) replaced by \( \phi^{p/q} \) and the latter is of prime period. In the following proof \( p \) is taken to be prime.

Assume on the contrary that there is a point \( Q \in E^{n+1} - F(S^n \times I) \). By (a), \( Q \in S^n \) and by a previous remark there is a retraction \( r: E^{n+1} \to \{ Q \} \to S^n \). Regarding \( F \) as a map into \( E^{n+1} - Q \) we would have \( rF_0 \) homotopic to \( rF_1 \). Now \( rF_0 \) is the identity map of \( S^n \) (hence of degree 1) while \( rF_1 \) has the property that \( (rF_1)\phi = rF_1 \) on account of condition (b).

To conclude the proof it shall be shown that any map \( g: S^n \to S^n \) satisfying \( g\phi = g \) has a degree divisible by \( p \).

By [2] there is a cycle of the form \( c + \phi(c) + \phi^2(c) + \cdots + \phi^{p-1}(c) \) in a generator of \( H^n(S^n, J_p) \). Calling this cycle \( z \) we have \( g(z) = pc = 0 \mod p \). Thus the degree of \( g \) is divisible by \( p \). This concludes the proof of Proposition 2.

Forrester’s family of straight lines may be described by

\[ F: S^n \times I \to E^{n+1} \]

where \( F(P, t) \) is the point \( Q \) on the line segment joining \( P \) to \( \phi(P) \) such that \( PQ/P\phi(P) = t/2 \).

If the notion of homotopy is translated into the language of a continuous family of curves then Proposition 2 becomes:

**Proposition 2’.** If (1) \( \phi: S^n \to S^n \) satisfies the condition stated in Proposition 2 and (2) from each point \( P \) of \( S^n \) there begins one curve of
$E^{n+1}$ so that the curves beginning at $P$ and $\phi(P)$ have the same terminal point and (3) the parametrization of these curves depend continuously on $P$, then this family of curves fills $E^{n+1}$.

**Proposition 3.** Let $R^n$ refer to $n$ dimensional Euclidean space. If for each direction in $R^n$ there is given in a continuous manner precisely one straight line with that direction, then this family of lines fills $R^n$.

**Proof.** Compactify $R^n$ to $E^n$ by adding two points at infinity for each direction in $R^n$. Then apply Proposition 2 or 2' with $p = 2$.

**Proposition 4.** Let $A$ be a compact subset of $R^n$. A necessary and sufficient condition that $A$ be a convex set with the property that each support plane has precisely one contact point is that there exists a continuous choice function on the set of $n-1$ dimensional planes, meeting $A$, with values in $A$. Moreover any such function is onto $A$.

**Proof.** Let $A$ be a convex subset of $R^n$ with the property that each plane of support has one point of contact. Assign to each cross-section its centroid. By Proposition 2 with $p = 2$ it is easy to show that this function is onto $A$ (compare to p. 13 of [3]).

The proof of sufficiency is left to the reader. In a subsequent paper the intersection properties of families of curves will be considered.

**Bibliography**

