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A THEOREM ON HYPERSIMPLE SETS

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Introduction. E. L. Post proved in his paper *Recursively enumerable sets of positive integers and their decision problems*[1] that a creative set cannot be reducible to a hypersimple set by truth tables (pp. 308–310). The present paper is concerned with the question whether a creative set can be Turing reducible to a hypersimple set. It will be shown that the answer to this question is affirmative. In fact, we shall establish the following more general statement: *for every recursively enumerable, but not recursive, set α, a hypersimple set β can be effectively found such that α and β are Turing reducible to each other.*

Preliminaries. A non-negative integer is called a *number*, a collection of numbers is called a *set* and a collection of sets a *class*. A set is *immune*, if it is infinite, but has no infinite recursively enumerable subset. For the definitions of a discrete array and a hypersimple set we refer to our paper *Two notes on recursively enumerable sets*[2] (p. 497). We shall use the following notations: "$\rho(f)$" for the range of the function $f(n)$, "E" for the class of all recursive sets, "F" for the class of all recursively enumerable (r.e.) sets, "α t-red β" for α is reducible to β by truth tables, and "α red β" for α is Turing reducible to β. If $a(t_1)$ and $a(t_2)$ are elements of the sequence $\{a(n)\}$, we say that $a(t_2)$ *succeeds* $a(t_1)$ in $\{a(n)\}$ if $t_2 > t_1$.

Definition. Let $a(n)$ be a 1-1 function. The element $a(t)$ of the

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[1] Bull. Amer. Math. Soc. vol. 50 (1944) pp. 284–316.

[2] Proc. Amer. Math. Soc. vol. 4 (1953) pp. 495–501.

sequence $\{a(n)\}$ is called *minimal* if $a(t) < a(t+k)$ for every $k \geqq 1$. The set of all elements in $\{a(n)\}$ which are not minimal is denoted by "hyp $a(n)$."

DEFINITION. The subset σ of α is called *hypersimple relative to* α if σ is r.e., $\alpha - \sigma$ infinite, and σ includes at least one row of every discrete array all of whose elements belong to α.

Let $a(n)$ be a 1-1 recursive function. It is now readily verified that the subset σ of $\rho(a)$ is hypersimple relative to $\rho(a)$ if and only if $a^{-1}(\sigma)$ is hypersimple.

1. The main result.

THEOREM 1. *For every set* $\alpha \in F - E$ *one can effectively find a r.e. set* β *such that*: (1) β *is hypersimple*, (2) β *t*-red α, (3) α red β.

PROOF. Let $a(n)$ be a 1-1 recursive function ranging over α, $\zeta =_{df}$ hyp $a(n)$ and $\beta =_{df} a^{-1}(\zeta)$. We claim that β satisfies the requirements.

(1) It suffices to show that ζ is hypersimple relative to α. We shall call an element of α minimal if it is minimal relative to $\{a(n)\}$. For every number k we can effectively list the elements in $a(0), \cdots, a(k)$ which are succeeded by a smaller element in this finite sequence. Since we can do this for $k = 1, 2, \cdots$, we conclude that ζ is r.e. Suppose $a(k_0)$ is the smallest element of α, then $a(k_0)$ is minimal; if $a(k_1)$ is the smallest element of $\alpha - \{a(0), \cdots, a(k_0)\}$, $a(k_1)$ is also minimal and $a(k_0) < a(k_1)$. Continuing this procedure we see that α contains infinitely many minimal elements, i.e., that $\alpha - \zeta$ is infinite. Let $\{\delta_n\}$ be a discrete array all of whose elements belong to α. Then there exist recursive functions $d(n, x)$ and $e(n)$ such that for every n, $\delta_n = \{d(n, 0), \cdots, d(n, e(n))\}$; we may assume without loss of generality that $d(n, x)$ is strictly increasing for every n. We know that α is not recursive. To prove that ζ includes at least one row of $\{\delta_n\}$ it suffices therefore to show that the assumption that ζ includes no row of $\{\delta_n\}$ implies that α is recursive. This can be done in the following manner. Suppose ζ included no row of $\{\delta_n\}$. Then every row of $\{\delta_n\}$ would contain at least one minimal element. Observe that $d(n, 0)$ is a 1-1 recursive function of n such that $d(n, 0) = \min \delta_n$ for every n. Let x be any number. Find the first element in $\{d(n, 0)\}$ which is greater than x, say $d(r, 0)$. Then δ_r contains a minimal element, say $d(r, s)$. Thus $x < d(r, 0) \leqq d(r, s) \leqq d(r, e(r))$. We can now locate the elements of δ_r in $\{a(n)\}$, i.e., we can effectively find the numbers $n_0, \cdots, n_{e(r)}$ such that $d(r, 0) = a(n_0), \cdots, d(r, e(r)) = a(n_{e(r)})$. Let $t_x =_{df} \max (n_0, \cdots, n_{e(r)})$. The element $a(t_x)$ is now

either equal to the minimal element $d(r, s)$ or it succeeds $d(r, s)$ in $\{a(n)\}$. It now follows from the fact that $d(r, s) > x$ that all elements in $\{a(n)\}$ which succeed $a(t_x)$ in $\{a(n)\}$ are greater than x. Hence $x \in \alpha$ if and only if $x \in \{a(0), \cdots, a(t_x)\}$. Since t_x is a recursive function of x, we would now possess an effective procedure to test whether x belongs to α. Thus α would be recursive.

(2) Let x be any number. Then $x \in \beta$ if and only if $a(x)$ is not minimal. Let $\alpha_x^* =_{df} \{0, \cdots, a(x)-1\} - \{a(0), \cdots, a(x-1)\}$, then $a(x)$ is not minimal if and only if α_x^* contains at least one element of α. There now exist recursive functions $a_{x,n}^*$ and $m(x)$ such that:

(a) if α_x^* is empty, then $a_{x,0}^* = 0$ and $m(x) = 0$,

(b) if α_x^* is nonempty, then

$$\alpha_x^* = \{a_{x,0}^*, \cdots, a_{x,m(x)}^*\}.$$

Let T_x be the truth table of order $m(x)+1$ whose last column is $-, -$ in case α_x^* is empty and is $+, +, \cdots, +, -$ in case α_x^* is nonempty. The decision problem of β is now reducible to that of α by the reduction which associates with every number x the finite sequence $a_{x,0}^*, \cdots, a_{x,m(x)}^*$ and the truth table T_x. Thus β t-red α.

(3) Let x be any number. We can effectively find a recursive function h_n (depending on x) such that all elements of α which are greater than x are generated in the sequence $\{a(h_n)\}$. We now raise the following questions about ζ:

$$\text{“}a(h_0) \in \zeta?\text{,”} \qquad \text{“}a(h_1) \in \zeta?\text{,”} \cdots .$$

After a finite number of questions the correct answer must be negative, since $\alpha - \zeta$ is infinite and all but finitely many elements of α are assumed by $a(h_n)$. Let $p(x)$ be the first n such that $a(h_n) \notin \zeta$. Then we know for every x that $a(h_{p(x)})$ is a minimal element of α greater than x. Thus $x \in \alpha$ if and only if $x \in \{a(0), \cdots, a(h_{p(x)}-1)\}$. Let $c_\alpha(x)$, $c_\beta(x)$, $c_\zeta(x)$ be the characteristic functions of α, β, ζ respectively. We have proved that $c_\alpha(x)$ is recursive in $c_\zeta(a(x))$. But $c_\zeta(a(x)) = c_\beta(x)$. It follows that $c_\alpha(x)$ is recursive in $c_\beta(x)$, i.e., that α red β.

REMARK 1. In the proof of (1) no use is made of the fact that there exists a hypersimple set. As a side result we have therefore a new existence proof for hypersimple sets.

REMARK 2. While the fact that $\alpha = \rho(a)$ is not recursive plays an essential role in the proof of (1), it is not used in the proof of (2). The statement β t-red α is therefore still valid if α is recursive. Thus, if α is recursive, β is recursive. This can also be seen in the following way. If $a(n)$ is a 1-1 recursive function, ranging over the recursive set α, we can decide for any n whether $a(n)$ is minimal in $\{a(n)\}$ by locating $a(n)$ in the recursive enumeration of α according to size.

COROLLARY 1. *There exist two r.e. sets* σ *and* τ *such that* σ t-red τ *is false, but* σ red τ *true.*

PROOF. Let σ be any creative set, $s(n)$ a 1-1 recursive function ranging over σ, and $\tau =_{df} s^{-1}$ (hyp $s(n)$). Then σ red τ is true; but σ t-red τ is false, because a creative set cannot be reducible to a hypersimple set by truth tables.

COROLLARY 2. *There exist two sets in* $F-E$ *which are not Turing reducible to each other if and only if there exist two hypersimple sets which are not Turing reducible to each other.*

2. A generalization.

DEFINITION. Let $a(n)$ be a 1-1 function and t any number. Then the *deficiency* dfc $a(t)$ of the element $a(t)$ in $\{a(n)\}$ is the number of elements which while smaller than $a(t)$ succeed $a(t)$ in $\{a(n)\}$.

Let $a(n)$ be a 1-1 recursive function. It is now easily verified that the function dfc $a(n)$ is recursive if and only if the set $\rho(a)$ is recursive. For let $\rho(a)$ be recursive. Then the statements

$$0 \in \rho(a),\ 1 \in \rho(a),\ \cdots,\ a(n) - 1 \in \rho(a)$$

can be effectively tested for every n; the results can then be compared with the elements $a(0), \cdots, a(n-1)$. Thus dfc $a(n)$ is recursive. To establish the converse, suppose $\rho(a)$ is not recursive. Then the set λ of all elements in $\{a(n)\}$ which have deficiency 0 (i.e., which are minimal) is immune by Theorem 1. But, if dfc $a(n)$ were recursive, λ would be r.e. We conclude that dfc $a(n)$ is not recursive.

THEOREM 2. *Let* $a(n)$ *be a* 1-1 *recursive function such that* $\alpha = \rho(a) \in F-E$. *Let* k *be any number* $\geqq 1$ *and* $\beta_k =_{df} \{n \mid \text{dfc } a(n) \geqq k\}$. *Then we know*: (1) β_k *is hypersimple*, (2) β_k t-red α, (3) α red β_k.

PROOF. Let $\zeta_k =_{df} a(\beta_k)$. It is now easily verified that ζ_k is r.e.

(1) It suffices to show that ζ_k is hypersimple relative to α. This can be done by induction on k. ζ_1 is hypersimple relative to α. The set $\alpha - \zeta_k$ is infinite, since it includes the infinite set $\alpha - \zeta_1$. Suppose ζ_{k^*} is hypersimple relative to α, where $k^* \geqq 1$. Assume that ζ_{k^*+1} were not hypersimple relative to α. Then there would exist a discrete array of elements in α, none of whose rows would be included in ζ_{k^*+1}, say $\{\delta_n\}$. Each row of $\{\delta_n\}$ would therefore contain at least one element with a deficiency $\leqq k^*$. We claim that $\{\delta_n\}$ must have infinitely

many rows containing at least one element with deficiency k^*; for otherwise $\{\delta_n\}$ would have a discrete subarray each row of which would contain at least one element with a deficiency $\leqq k^*-1$; this would contradict the induction hypothesis that ζ_{k^*} was hypersimple relative to α. Since $\{\delta_n\}$ has infinitely many rows which contain at least one element with deficiency k^*, we can, by comparing $\{\delta_n\}$ with a recursive enumeration of ζ_{k^*}, obtain a subarray of $\{\delta_n\}$ each row of which contains only elements with a deficiency $\geqq k^*$ and at least one element with deficiency k^*. Let $\{\rho_n\}$ be such a subarray of $\{\delta_n\}$. We now transform ρ_0 into σ_0 in the following manner:

(a) replace every element x of ρ_0 by the first element in $\{a(n)\}$ which while smaller than x succeeds x,

(b) omit all repetitions which might arise because different elements of ρ_0 are replaced by the same element.

Let $t =_{df} \max \{n \mid a(n) \in \sigma_0\}$. Only finitely many rows of $\{\rho_n\}$ can contain at least one of the elements $a(0), \cdots, a(t)$; thus we can effectively find the first row of $\{\rho_n\}$ all of whose elements succeed $a(t)$ in $\{a(n)\}$, say ρ_i. We now define σ_1 in terms of ρ_i as we defined σ_0 in terms of ρ_0. Continuing this procedure we transform the discrete array $\{\rho_n\}$ into a discrete array $\{\sigma_n\}$. It now follows from the properties of $\{\rho_n\}$ and the definition of $\{\sigma_n\}$ that every row of $\{\sigma_n\}$ contains at least one element with a deficiency $\leqq k^*-1$. Thus ζ_{k^*} would include no row of the discrete array $\{\sigma_n\}$ of elements in α; this would contradict the assumption that ζ_{k^*} is hypersimple relative to α. We conclude that ζ_{k^*+1} is also hypersimple relative to α.

(2) $x \in \beta_k$ if and only if $a(x)$ has a deficiency $\geqq k$. We now define α_x^*, $a_{x,n}^*$, and $m(x)$ in the same way as in the proof of the second part of Theorem 1. Let $T_{k,x}$ be the truth table of order $m(x)+1$ such that: if α_x^* is empty the last column of $T_{k,x}$ is $-, -$, if α_x^* is nonempty the last column of $T_{k,x}$ contains the symbol "$+$" in the tth row if and only if the truth assignment in the tth row contains the symbol "$+$" at least k times. The decision problem of β_k is now reducible to that of α by the reduction which associates with every number x the finite sequence $a_{x,0}^*, \cdots, a_{x,m(x)}^*$ and the truth table $T_{k,x}$. Thus β_k t-red α.

(3) We apply again induction on k. We know that α red β_1. Suppose α red β_{k^*}, where $k^* \geqq 1$. The fact that $\zeta_{k^*+1} \subset \zeta_{k^*}$ implies that $\beta_{k^*+1} \subset \beta_{k^*}$. Suppose $\beta_{k^*} - \beta_{k^*+1}$ is finite. Then α red β_{k^*+1} follows from α red β_{k^*}, because β_{k^*} is many-one reducible to β_{k^*+1}. Now suppose $\beta_{k^*} - \beta_{k^*+1}$ is infinite, then $\zeta_{k^*} - \zeta_{k^*+1}$ is also infinite. We can effectively find a 1-1 recursive function c_n such that ζ_{k^*} is generated in the sequence $a(c_0), a(c_1), \cdots$. We now raise the following questions about ζ_{k^*+1}:

(I) $\quad$ "$a(c_0) \in \zeta_{k^*+1}$?", $\quad$ "$a(c_1) \in \zeta_{k^*+1}$?", $\cdots$.

Among the correct answers infinitely many must be negative, since $\zeta_{k^*} - \zeta_{k^*+1}$ is infinite. Let $\{d_n\}$ be the sequence of all m for which there exists an n such that $m = a(c_n) \notin \zeta_{k^*+1}$ ordered in the same way as they occur in (I). Observe that for every n the deficiency of d_n is exactly k^*. Let e_n be the minimum of the k^* elements in $\{a(n)\}$ which while smaller than d_n succeed d_n. Now suppose x is any number. Since there exists an n_x such that $\{a(n)\}$ contains only elements $>x$ after $a(n_x)$, there must be a number $t(x)$ such that $e_{t(x)} > x$. But $e_{t(x)}$ is minimal, for the definition of e_n implies that e_n is minimal for every n. Hence $x \in \alpha$ if and only if $x \in \{a(0), a(1), \cdots, e_{t(x)}\}$. The number $e_{t(x)}$ can be obtained after finitely many questions in (I) are correctly answered. Let $f(x)$, $g(x)$, $h(x)$ be the characteristic functions of α, β_{k^*+1}, ζ_{k^*+1}, respectively. We have proved that $f(x)$ is recursive in $h(a(x))$. But $h(a(x)) = g(x)$. It follows that $f(x)$ is recursive in $g(x)$, i.e., that α red β_{k^*+1}.

Corollary. *Let $a(n)$ be a 1-1 recursive function such that $\rho(a)$ is not recursive. Let k be any number $\geqq 1$. Then:*

(1) *there are infinitely many elements in $\{a(n)\}$ with a deficiency $\geqq k$,*

(2) *the set of all elements in $\{a(n)\}$ with a deficiency $<k$ is immune,*

(3) *the set of all elements in $\{a(n)\}$ with deficiency k is finite or immune.*

Proof. The first two parts follow immediately from the fact that ζ_k is hypersimple relative to $\rho(a)$. The third part follows from the second part, because the set of all elements with deficiency k is included in the set of all elements with a deficiency $<k+1$.

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