A RADICAL FOR NEAR-RINGS

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1. **Introduction.** In a recent paper [1], D. W. Blackett extended a number of results from the theory of semisimple rings to semisimple near-rings. Whereas a semisimple ring is ordinarily defined as being a ring from which the (or a) radical has been "deleted," in [1] a semisimple near-ring was defined without a radical-like ideal being introduced first. It is the purpose of this note to demonstrate the existence of an ideal R in a near-ring N such that the difference near-ring $N-R$ is semisimple (in the sense of Blackett). The ideal R will be the zero ideal if and only if N is semisimple. Furthermore, if N is a ring, then R will be the usual nilpotent radical. Thus it seems appropriate to call R the radical of N.

Here, as in [1], attention is restricted to those near-rings which satisfy the descending chain condition for right modules and the requirement that the zero element of the near-ring annihilates the near-ring from the left. Terminology not defined here may be found in [1].

2. **Definition of R.** Let M be a minimal right module of N, and let $r(M)$ be the set of all elements of N which annihilate M from the right. Clearly $r(M)$ is an ideal (two-sided) of N. Now let $\mathfrak{M}$ be the set of all minimal right modules of N and define S to be intersection of all the $r(M)$, $M\in\mathfrak{M}$. Therefore S is an ideal of N.

THEOREM 1. *N is semisimple if and only if S is the zero ideal.*

Let S be the zero ideal. Now N was defined to be semisimple if it contained no nilpotent right modules [1], so let Q be a nilpotent right module of N. It may be assumed without loss of generality that Q is a minimal right module of N. Since $Q^2=0$ and since $S=0$, there exists another minimal right module of N, say P, such that $PQ\neq 0$. Hence there exists an element $p\in P$ such that $pQ=P$. But since $pQ^2=0=PQ$, we see that N is semisimple.

Now suppose N is semisimple. Since S is an ideal of N it is a right module and so contains P, a minimal right module of N. But $S\subseteq r(P)$ so that $P^2=0$, and it follows that S must be the zero ideal.

THEOREM 2. *The near-ring $N_1=N-S$ is semisimple.*

Let N_1 contain the nilpotent minimal right module Q_1, and let Q

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be the pre-image of Q_1 in N. Then Q is a right module of N, and $Q^2 \subseteq S$. If $Q \not\subseteq S$, then there is a minimal right module M_1 of N such that $M_1 Q \neq 0$. Hence M_1 contains an element m such that $mQ = M_1$. But $M_1 Q = (mQ)Q \subseteq mS = 0$, and so $Q \subseteq S$. Therefore $Q_1 = 0$ and N_1 is semisimple.

It will be illustrated later by a simple example that S may be a larger ideal than desired. Consequently the following definition of R must be given.

First, let $\mathfrak{P}$ be the set of all ideals P of N which contain no nonzero idempotents but which contain at least one nonzero element which annihilates all idempotents of N from the left. Then define T to be the sum of all $P \in \mathfrak{P}$ (that is, the set of all sums of a finite number of elements chosen arbitrarily from the ideals P). If $\mathfrak{P}$ is void, define T to be the zero ideal.

Now form S, as previously defined, in the near-ring $N_1 = N - T$, and let $\overline{N} = N_1 - S$. Then we define R, the radical of N, to be the kernel of the homomorphism $\sigma: N \rightarrow \overline{N}$.

THEOREM 3. *The near-ring $N - R$ is semisimple, and N is a semisimple near-ring if and only if R is the zero ideal.*

This is an immediate consequence of the preceding results and the structure theory for semisimple near-rings [**1**].

3. The ring case.

To justify calling R the radical of N, the following is necessary.

THEOREM 4. *If N is a ring, then R coincides with the usual radical.*

The chain condition for right modules becomes the chain condition for right ideals when N is a ring. Thus

$$\begin{aligned} N &= e_1 N + \cdots + e_n N + r(e) \\ &= N e_1 + \cdots + N e_n + l(e), \end{aligned}$$

where $e_i N$ and $N e_i$ are indecomposable right and left ideals, the e_i are primitive idempotents, e is the sum of the e_i, and $r(e)$ and $l(e)$ are the sets of all elements of N which annihilate e from the right and left respectively. Therefore T is nilpotent and $N_1 = N - T$ contains a right identity element. Clearly the ideal S of N_1 is also nilpotent, so that if K is the usual radical of N, $R \subseteq K$. Since $N - R$ contains no nilpotent right ideals, $R \supseteq K$. Hence $R = K$.

It should be pointed out that this definition of R is essentially an adaptation of the arithmetic characterization of the radical of a ring with identity element given by N. Jacobson [**3**].

REMARK 1. The necessity of introducing the ideal T is demonstrated by the following example. Let N be the ring having basis elements a and b over the rational field, where multiplication is defined as follows: $a^2=a$, $ab=b$, $ba=b^2=0$. The only minimal right ideal of N is the principal ideal (b), and therefore $S=r(b)=N$. However, $R=(b)$.

REMARK 2. It was recently pointed out to the author that the ideal S was very similar to the definition of a radical for rings given by O. Goldman [2]. However this similarity is somewhat superficial due to a perhaps unfortunate choice of terminology. A right module, as defined in [1], becomes nothing more than a right ideal if N is a ring, whereas a right module in ring theory is considerably more general than a right ideal. Hence our Theorem 4 is not a consequence of Goldman's work.

BIBLIOGRAPHY

1. D. W. Blackett, *Simple and semisimple near-rings*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 772–785.

2. O. Goldman, *A characterization of semisimple rings with the descending chain condition*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 1021–1027.

3. N. Jacobson, *The theory of rings*, Mathematical Surveys, no. 2, New York, American Mathematical Society, 1943, p. 66.

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