ON THE MINIMUM MODULUS OF A ROOT OF A POLYNOMIAL

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Landau showed [1][1] that the equation $1+z+\alpha z^m=0$, $m>1$, has at least one root with the modulus $\leqq 2$ and that the equation

$$1 + z + \alpha z^m + \beta z^n = 0 \qquad (1 < m < n)$$

has at least one root with the modulus $\leqq 17/3$. He posed the problem whether the equation

$$P_K(z) = 1 + z + \alpha_1 z^{n_1} + \cdots + \alpha_{K-1} z^{n_{K-1}} = 0$$

has at least one root with a modulus not greater than a number $M(K)$, depending only on the number of terms $P_K(z)$ and not at all on the numbers $\alpha_1, \alpha_2, \cdots, \alpha_{K-1}, n_1, n_2, \cdots, n_{K-1}$.

This problem was solved by P. Montel. Montel [2] in his paper, written in 1923, showed that the number $M(K)$ has the simple value K, and that whenever the root assumes this maximum value K all the roots of the polynomial are equal to $-K$.

In this note we establish the following stronger result.

THEOREM. *The equation* $P_K(z)=1+z+\alpha_1 z^{n_1}+\cdots+\alpha_{K-1}z^{n_{K-1}}=0$ $(2\leqq n_1<n_2<n_3<\cdots<n_{K-1})$, $\alpha_i\neq 0$ $(i=1, 2, \cdots, K-1)$ *has at least one root within or on the circumference of a circle with the center* $-\lambda/2$ *and radius* $\lambda/2$ *where* $\lambda=(n_1/(n_1-1))\cdot(n_2/(n_2-1))\cdots(n_{K-1}/(n_{K-1}-1))$.

PROOF. To prove the theorem and simplify operations let us consider the equation $f(z)=1+z+\alpha z^m+\beta z^n+\gamma z^p=0$ $(2\leqq m<n<p)$. Putting $z=1/\delta$ we have $\phi(\delta)=\delta^p+\delta^{p-1}+\alpha\delta^{p-m}+\beta\delta^{p-n}+\gamma=0$; the derivative may be written $\phi'(\delta)=\delta^{p-n-1}\cdot\phi_1(\delta)$ where

$$\phi_1(\delta) = p\cdot\delta^n + (p-1)\delta^{n-1} + \alpha(p-m)\delta^{n-m} + \beta(p-n).$$

Similarly we write $\phi_1'(\delta)=\delta^{n-m-1}\cdot\phi_2(\delta)$ where $\phi_2(\delta)=p\cdot n\delta^m+(p-1)\cdot(n-1)\delta^{m-1}+\alpha(p-m)(n-m)$. Similarly we write $\phi_2'(\delta)=p\cdot mn\delta^{m-2}\cdot(\delta+\theta)$ where $\theta=(p-1)/p\cdot(n-1)/n\cdot(m-1)/m$.

Let $\Pi(x)$ be the half-plane $R(\delta)\geqq x$. If $0>x>-\theta$, $\phi_2(\delta)$ must have at least one root not in $\Pi(x)$, since otherwise by Lucas' Theorem [3] all the roots of $\phi_2'(\delta)$ would be in $\Pi(x)$, and this contradicts the fact that $\delta=-\theta$ is a root of $\phi_2(\delta)$. Similarly $\phi_1(\delta)$ must have at least one

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[1] Numbers in brackets refer to the list of references.

root not in $\Pi(x)$, and hence $\phi(\delta)$ must have at least one root not in $\Pi(x)$. But $\phi(\delta)$ has only a finite number of roots, so that if all these roots were in $R(\delta) > -\theta$ one could select an $x > -\theta$ so that all the roots would be in $\Pi(x)$. Hence $\phi(\delta)$ has at least one root in $R(\delta) \leqq -\theta$, and therefore $f(z)$ has at least one root in the image of this half-plane under $1/z$, namely the circle $|z+1/2\theta| \leqq 1/2\theta$. It is easy to see that in the general case $1/\theta$ is replaced by λ and this completes the proof of the theorem.

Since $\lambda/2 \leqq K/2$ where $K+1$ is the number of the terms of the equation

$$P_K(z) = 1 + z + \alpha_1 z^{n_1} + \cdots + \alpha_{K-1} z^{n_{K-1}} = 0,$$

and since the circle of centre $-\lambda/2$ and radius $\lambda/2$ is covered by the circle of center $-K/2$ and radius $K/2$, we have the theorem:

THEOREM. *The polynomial* $P_K(z) = 1 + z + \alpha_1 z^{n_1} + \cdots + \alpha_{K-1} z^{n_{K-1}}$ *of* $K+1$ *terms has at least one root within or on the circumference of a circle of centre* $-K/2$ *and radius* $K/2$.

COROLLARY. *The equation* $\alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_m z^m = 0$ *has at least one root within or on the circumference of a circle of centre* $-(\alpha_0/\alpha_1) \cdot m/2$ *and radius* $|\alpha_0/\alpha_1| \cdot m/2$.

If in the proof of the main theorem, the plane $\Pi(x)$ is replaced by any closed half-plane containing the origin but not containing $-\theta$, the reasoning is still valid and one obtains as a result the following:

THEOREM. *Let C be a closed circular disk containing on its boundary the points* $z=0$ *and* $z=-\lambda$. *Then C contains at least one root of* $P_K(z)$.

REFERENCES

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3. Morris Marden, *The geometry of the zeros of a polynomial in a complex variable*, 1949, pp. 14–15.

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