MEDIANS AND BETWEENNESS

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In a recent paper [4] it is shown that lattices and trees have a common generalization. These systems will be called median semilattices. These semilattices are characterized below by means of segments (§§2 and 4) and by means of betweenness (§§3 and 4). In §5 distributive lattices are characterized by means of medians with no assumption made as to the existence of universal bounds (compare Problem 66 of [1]). In another paper [5], median semilattices are characterized in terms of a binary operation and it is shown that these systems can be imbedded in distributive lattices.

1. Medians. Consider a set S closed under a ternary operation \((a, b, c)\), called the median of \(a\), \(b\), and \(c\), satisfying the identities:
   \[\begin{align*}
   (M) & \quad (a, a, b) = a. \\
   (N) & \quad ((a, b, c), (a, b, d), e) = ((c, d, e), a, b).
   \end{align*}\]
   We call \(S\) a median semilattice.

   We use freely properties previously derived for the median [4, §8]. In particular we note \((a, b, c)\) is invariant under permutations of \(a\), \(b\), and \(c\). We say \(x\) is between \(a\) and \(b\) and write \(axb\) if and only if \(x = (a, x, b)\). The segment \((a, b)\) is defined as the set of elements between \(a\) and \(b\). It is immediate that \((a, a) = \{a\}\) and that \((a, b) = (b, a)\) contains \(a\) and \(b\).

   Theorems 1.1, 1.2, and 1.3 show that Postulates M and N imply Postulates P, Q, and R of the next section.

   \((1.1)\) \((a, b) \cap (a, c) = (a, d)\) where \(d = (a, b, c)\).
   \[\text{Proof.}\] Let \(x = (a, x, b) = (a, x, c)\). Then \(x = (x, x, a) = ((a, x, b), (a, x, c), a) = ((b, c, a), a, x) = (a, x, d)\). Conversely, let \(x = (a, x, d)\). From [4, (8.4) and (8.6)], \(d = (a, d, b)\) and \(x = (a, x, (a, d, b)) = (a, (b, d, x)) = (a, b, x)\). Similarly, \(x = (a, x, c)\).

   \((1.2)\) \((a, b) \subseteq (a, c)\) implies for all \(x\) that \((x, a) \cap (x, c) \subseteq (x, b)\).
   \[\text{Proof.}\] We note \(b = (a, b, c)\). Let \(r = (a, r, x) = (c, r, x)\). Then \(r = (r, r, b) = ((x, r, a), (x, r, c), b) = ((a, c, b), x, r) = (b, x, r)\).

   \((1.3)\) \((a, b) \cap (a, c) = \{a\}\) implies \(a \subseteq (b, c)\).
   \[\text{Proof.}\] It is given that \(r = (a, r, b) = (a, r, c)\) implies \(r = a\). It follows, from [4, (8.4)], that \((b, a, c) = a\).
Theorems 1.4 and 1.5 are for future reference. The latter theorem is of some intrinsic interest.

(1.4) \( abc \cdot adc \cdot bxd \rightarrow cxa \).

**Proof.** Since \( b = (a, b, c) \) and \( d = (a, d, c) \), \( x = (b, x, d) = ((a, b, c), x, (a, d, c)) = (a, c, (b, x, d)) = (a, c, x) \).

(1.5) \( (a, x, b) = (a, y, b) \) if and only if \( (x, a, y) = (x, b, y) \).

**Proof.** By symmetry, it is sufficient to prove necessity. Let \( z = (x, b, y) \) where \( (a, x, b) = (a, y, b) \). Then \( (x, a, y) = ((y, b, y), a, x) = ((a, x, y), (a, x, b), y) = ((a, x, y), (a, y, b), y) = (z, a, y) \). Similarly \( (x, a, y) = (z, a, x) \). Hence \( z = (z, z, a) = ((x, y, b), z, a) = ((z, a, x), (z, a, y), b) = (x, a, y) \).

2. **Median segments.** Consider a set \( S \) of elements \( a, b, c, \ldots \). The symbol \( (a, b) \), called the segment from \( a \) to \( b \), represents a subset of \( S \) determined by the ordered pair \( a, b \). These segments have as properties:

(P) Given \( a, b, \) and \( c \), there exists an element \( d \) such that \( (a, b) \cap (a, c) = (a, d) \).

(Q) \( (a, b) \cap (a, c) = (a, b) \) implies \( (x, a) \cap (x, c) \subset (b, x) \) for all \( x \).

(R) \( (a, b) \cap (a, c) = (a, a) \) implies \( (b, a) \cap (b, c) = \{a\} \).

We say \( b \) is between \( a \) and \( c \) and write \( abc \) if and only if \( b \in (a, c) \). Theorems 2.2, 2.8, and Postulate Q show that \( S \) satisfies Postulates B1, D1, and I of the next section.

The following result is obtained from (Q) by choosing \( a = b = c \).

(2.1) \( (a, b) = (b, a) \).

From (R), with \( a = b = c \), we have

(2.2) \( (a, a) = \{a\} \).

(2.3) Both \( a \) and \( b \) belong to \( (a, b) \).

**Proof.** Let \( (a, b) \cap (a, a) = (a, d) \). Since \( (a, d) \subset (a, a) \), we have from (Q) that \( (a, a) \cap (a, b) \subset (a, d) \) and hence \( a \in (a, a) \subset (a, d) \subset (a, b) \). That \( b \in (a, b) \) now follows from (2.1).

(2.4) \( (a, b) \cap (a, c) \subset (a, d) \) implies that \( (a, d) \cap (b, c) \subset (d, d) \).

**Proof.** We apply (Q) twice. From \( (a, d) \subset (a, b) \) we have \( (c, a) \cap (c, b) \subset (c, d) \). From \( (a, d) \subset (a, c) \), \( (d, a) \cap (d, c) \subset (d, d) \). Thus \( (a, d) \cap (b, c) = (a, d) \cap (a, c) \cap (b, c) \subset (a, d) \cap (c, d) \subset (d, d) \).

(2.5) The relations \( b \in (a, c) \), \( (a, b) \subset (a, c) \), and \( (a, b) \cap (b, c) = (b, b) \) are equivalent.

**Proof.** Assume \( b \in (a, c) \) and let \( (a, b) \cap (a, c) = (a, d) \). Then from (2.3) and (2.4), \( b \in (a, b) \cap (a, c) \cap (b, c) \subset (d, d) \). Hence \( b = d \) and \( (a, b) = (a, d) \subset (a, c) \). Next, if \( (a, b) \subset (a, c) \), \( (b, a) \cap (b, c) \subset (b, c) \). Since \( b \in (b, a) \cap (b, c) \), \( (b, a) \cap (b, c) = \{b\} \). Finally if \( (b, a) \cap (b, c) = (b, b) \), we have from (R) that \( b \in (a, c) \).

We have as a corollary
(2.6) \((a, b) = (a, c)\) implies \(b = c\).

(2.7) If \((a, b) \cap (a, c) = (a, d)\), then \((a, d) \cap (b, c) = \{d\}\).

Proof. From \(d \in (a, d)\), we have \(d \in (b, a)\) and \(d \in (c, a)\). Hence \((b, d) \subseteq (b, a), (c, d) \subseteq (c, a)\), and it follows that \((b, d) \cap (c, d) \subseteq (a, d)\). Moreover, we have \((b, d) \cap (d, a) = (d, d) = (c, d) \cap (d, a)\). Hence \((d, d) = (a, d) \cap (b, d) \cap (c, d) = (b, d) \cap (c, d)\), and \(d \in (b, c)\). This, with (2.4), proves the theorem.

The element \(d\), whose existence is given by (P) and whose uniqueness is implied by (2.6), is called the median of \(a, b, c\) and is denoted by \((a, b, c)\). As a corollary of (2.7) we have

\[
(2.8) (a, b) \cap (b, c) \cap (c, a) = \{d\}
\]

where \(d = (a, b, c)\).

From (2.1) and (2.8) it is clear that \((a, b, c)\) is invariant under permutations of \(a, b,\) and \(c\).

3. Median betweenness. Consider a set \(S\) in which there exists a ternary relation called betweenness. If \(b\) is between \(a\) and \(c\), we write \(abc\). This relation has the properties:

(D1) To \(a, b,\) and \(c\) corresponds an element \(w\) such that \(awb \cdot bwc\cdot cwa\).

(B1) \(aba \rightarrow a = b\).

(I) \(abc \cdot abd \cdot cxd \rightarrow xba\).

Setting \(a = c\) in (D1) we have from (B1) that

\[(3.1) \text{ Both } aab \text{ and } baa \text{ hold.}\]

Setting \(c = x = d\) in (I), we have

\[(3.2) abc \rightarrow cba\]

Setting \(a = x\) and \(b = d\) in (I), we have

\[(3.3) abc \cdot bac \rightarrow a = b\]

Since (I) is Transitivity T7 of [3], it follows from that paper (p. 102) that

\[(3.4) abc \cdot acd \rightarrow bcd\]

(3.5) Element \(w\) in (D1) is unique.

Proof. Assume \(awb \cdot bwc \cdot cxa\). By (I), \(awb \cdot awc \cdot bxc \rightarrow xwa\) and \(awb \cdot axc \cdot bwc \rightarrow wxa\). By (3.3), \(x = w\).

We call element \(w\) in (D1) the median of \(a, b,\) and \(c\) and write \(w = (a, b, c)\). By (3.2), \((a, b, c)\) is invariant under permutations.

\[(3.6) abc \cdot acd \rightarrow abd\]

Proof. Let \(r = (a, b, d)\). We have \(bra \cdot brc \cdot adr \rightarrow crb, crb \cdot cba \rightarrow rba,\) and \(bra \cdot rba \rightarrow b = r\). Thus \(ard\) implies \(abd\).

\[(3.7) abc \cdot bcd \cdot ade \rightarrow ecb\]

Proof. Let \(r = (e, c, b)\). We have \(crb \cdot cba \rightarrow cra, cra \cdot cre \cdot ade \rightarrow drc, brc \cdot bcd \rightarrow rcd, cdr \cdot rcd \rightarrow r = e,\) and \(erb \rightarrow ecb\).

\[(3.8) \text{ Postulates } M \text{ and } N \text{ hold.}\]

Proof. (M) follows from (3.1) and (3.5). In proving (N) we let...
Using (I), (3.7), and (3.4) we have

\[\begin{align*}
\text{era} \cdot \text{crb} \cdot \text{aqb} & \rightarrow \text{qrc}, \\
\text{dsa} \cdot \text{dsb} \cdot \text{aqb} & \rightarrow \text{qsd}, \\
\text{tqa} \cdot \text{tgb} \cdot \text{arb} & \rightarrow \text{rgt}, \\
\text{tqa} \cdot \text{tgb} \cdot \text{asb} & \rightarrow \text{sqt}, \\
\text{dsq} \cdot \text{sql} \cdot \text{dta} & \rightarrow \text{cqs}, \\
\text{crq} \cdot \text{cqs} & \rightarrow \text{rqs}, \\
\text{epr} \cdot \text{eps} \cdot \text{rqs} & \rightarrow \text{qpe}, \\
\text{tqr} \cdot \text{tgs} \cdot \text{rps} & \rightarrow \text{qpt}, \\
\text{epq} \cdot \text{pqt} \cdot \text{etc} & \rightarrow \text{cqp}, \\
\text{epq} \cdot \text{pqt} \cdot \text{eld} & \rightarrow \text{dqp}, \\
\text{crq} \cdot \text{cqp} & \rightarrow \text{rqp}, \\
\text{dsq} \cdot \text{dqp} & \rightarrow \text{sqp}, \\
\text{pqr} \cdot \text{pqs} \cdot \text{rps} & \rightarrow \text{ppq}.
\end{align*}\]

Hence \( p = q \) and (N) is proved.

We may summarize the results of these sections by the following theorem.

(3.9) A median semilattice is characterized by (M, N), by (P, Q, R), or by (B₁, D₁, I).

4. **Further characterizations.** In this section we refer to some additional postulates:

- **(D)** To \( a, b, \) and \( c \) there corresponds a unique element \( w \) such that \( awb \cdot bwc \cdot cwa \).
  - (A) \( abc \rightarrow cba \).
  - (B₂) \( abc \cdot acb \leftarrow b = c \).
  - (E) \( abc \cdot acd \rightarrow dcb \).
  - (F) \( abc \cdot acd \rightarrow dba \).
  - (G) \( abc \cdot adc \cdot bxd \rightarrow cxa \).
  - (H) \( abc \cdot acd \cdot ade \rightarrow edb \).
  - (J) \( abc \cdot bcd \cdot ade \rightarrow ecb \).

Postulates A through J are variants respectively of Postulates \( \alpha, \beta, t₁, t₂, T₅, T₆, \) and \( T₄ \) of [3]. The following notation is used for sets of postulates: \( \Sigma₀ \) for (D, B₁, E), \( \Sigma₁ \) for (D, B₁, F), \( \Sigma₂ \) for (D, B₁, G), \( \Sigma₃ \) for (D, B₁, H), \( \Sigma₄ \) for (D₁, B₁, J), and \( \Sigma₅ \) for (D₁, B₁, I).

In the previous section we proved \( \Sigma₅ \) characterizes median semilattices. By that section and [4], \( \Sigma₅ \) implies Postulates D, E, F, and J.
By (1.4), $\Sigma_5$ implies (G). By [3], $\Sigma_5$ implies (H). We summarize as follows:

(4.1) $\Sigma_5$ implies $\Sigma_i$ for $i = 0, 1, 2, 3, 4$.

The following theorem holds since, by the previous section, the properties stated are consequences of (D_i) and (B_i).

(4.2) $\Sigma_i$ implies $aab \cdot baa$ for $i = 0, 1, 2, 3, 4$. Hence if $abc$ holds, element $w$ of (D_i) may be chosen as $b$.

(4.3) $\Sigma_i$ implies (A) for $i = 0, 1, 2, 3, 4$.

**Proof.** In (E), let $a = b$. In (F), let $b = c$. In (G), let $b = x = d$. In (H), let $a = b = c$. In (J), let $a = b$ and $c = d$.

(4.4) $\Sigma_i$ implies (B_2) for $i = 0, 1, 2, 3, 4$.

**Proof.** For $i = 0, 1, 2, 3$, this is a consequence of (4.2) and (D). For $\Sigma_4$, choose $a = d$ and $b = e$ in (J).

(4.5) $\Sigma_4$ implies (I).

**Proof.** By (4.3), (4.4), and [3] we have (E). To prove (I), assume $abc \cdot abd \cdot cxd$. By (D_i), for some element $r$, $arb \cdot brx \cdot xra$. We have $arb \cdot abc \rightarrow rbc$, $cbr \cdot brx \cdot cxd \rightarrow drb$, $arb \cdot abd \rightarrow rbd$, and $dbr \cdot drb \rightarrow r = b$. Thus $xra \rightarrow xba$.

In discussing $\Sigma_i$, $i = 0, 1, 2, 3$, we may denote element $w$ of (D) by $(a, b, c)$. By (4.3), $(a, b, c)$ is invariant under permutations.

(4.6) $\Sigma_2$ and $\Sigma_3$ each imply $\Sigma_1$.

**Proof.** By (4.3), (4.4), and [3], (G) implies (F), and (H) implies (E). It remains to show (H) implies (F). Assume $abc \cdot acd$. Let $r = (a, b, d)$. Then $arb \cdot abc \rightarrow rbc$, $arb \cdot abc \cdot acd \rightarrow dcr$, $dcr \cdot drb \rightarrow crb$, and $crb \cdot cbr \rightarrow b = r$. Thus $arb \rightarrow abd$.

(4.7) $\Sigma_1$ implies (E).

**Proof.** Assume $abc \cdot acd$. Let $x = (b, c, d)$. Then $dxc \cdot dca \rightarrow dxa$, $cxb \cdot cba \rightarrow cxa$, and $x = (a, c, d) = c$. Thus $bxd \rightarrow bcd$.

As in [4] it is convenient to extend the betweenness notation to a notation for chains. Thus $abcd$ denotes $abc \cdot abd \cdot acd \cdot bcd$. If $\Sigma_1$ holds, by repeated use of (E) and (F) we may obtain implications such as $abcd \cdot bxc \rightarrow abxcd$.

(4.8) If $\Sigma_1$ holds, $amp \cdot amq \cdot pmq \cdot mpq \cdot mpq \cdot pxq \rightarrow apx$.

**Proof.** Let $r = (a, p, x), s = (a, q, x), and t = (a, r, s)$. Since $atr \cdot arp \rightarrow atp$, $ats \cdot asq \rightarrow atq$, and $pxq \cdot prx \cdot xsq \rightarrow prtsq$, we have $t = (a, p, q) = m$. It follows that $amr \cdot arp \rightarrow mzp$ and $mzd \cdot mpx \cdot pxr \rightarrow mpzr \rightarrow r = p$. Finally $axr \rightarrow apx$.

(4.9) $\Sigma_1$ implies (I).

**Proof.** Given $abc \cdot abd \cdot cxd$. Let $m = (b, c, d), p = (m, c, x), and q = (m, d, x)$. Then $abc \cdot bmc \rightarrow amc, abd \cdot bmd \rightarrow amd$, and $m = (a, c, d)$. Thus $amc \cdot mpc \rightarrow amp$, and $amd \cdot mqd \rightarrow amq$. Also $cdx \cdot cpx \cdot xqd \rightarrow pxq$ and $cmd \cdot cpm \cdot mqd \rightarrow pmq$. Since $mpx$ and $mqx$ hold, we have by (4.8)
that \( apx \) holds. Finally \( abc \cdot bmc \cdot mpc \rightarrow abmpc \) and \( abp \cdot apx \rightarrow abx \).

As a consequence of (4.1), (4.5), (4.6), and (4.9), we have the following theorem.

(4.10) Median semilattices are characterized by \( \Sigma_i \) for \( i = 1, 2, 3, 4, 5 \).

A particularly simple characterization in terms of segments may be based on \( \Sigma_i \). Postulates used are:

(\( \alpha \)) Given \( a, b, \) and \( c \), there exists \( w \) such that \((a, b) \rangle (b, c) \rangle (c, a) = \{w\} \).

(\( \beta \)) \((a, a) = \{a\} \).

(\( \gamma \)) If \( b \in (a, c) \), then \((a, b) \subseteq (a, c) \rangle (c, a) \).

(4.11) Median semilattices are characterized by \((a, \beta, \gamma)\).

Proof. Let \( c = a \) in (\( \alpha \)). Since \( w \in (a, a) = \{a\}, w = a \). It follows that \( a \in (a, b) \rangle (b, a) \). Since \( a \in (b, a) \) and \( b \in (a, b) \), we have from (\( \gamma \)) that \((a, b) = (b, a) \). Clearly \((a, b) \subseteq (a, c) \rangle (c, a) \) implies \( b \in (a, c) \). It is now easy to prove (F) in the form: if \( b \in (a, c) \) and \( c \in (a, d) \), then \( b \in (a, d) \).

The five-element modular nondistributive lattice (see §6) shows that, in \( \Sigma_3 \), we cannot weaken (D) to (D\( _1 \)) and still obtain a median semilattice. As an example showing that we cannot weaken (D) in \( \Sigma_3 \) or in \( \Sigma_2 \) we have \( \{a, b, c, r, s\} \) where \( a r s \cdot b r s \cdot c r s \) and where both \( r \) and \( s \) are "medians" of \( a, b, \) and \( c \). It has not been determined whether \( \Sigma_0 \) characterizes median semilattices or not.

5. Distributive lattices. Consider a median semilattice \( S \) which satisfies the following postulate:

(O) \( S \) contains sequences \( u_1, u_2, \ldots \) and \( v_1, v_2, \ldots \) such that to each \( a \) in \( S \) there corresponds an ordinal \( n(a) \) with the property \((u_i, a, v_j) = a \) for \( i, j \geq n(a) \).

(5.1) To each pair of elements \( a \) and \( b \) there corresponds an ordinal \( n(a, b) \) such that for \( i \geq n(a, b) \) both \((a, u_i, b) \) and \((a, v_i, b) \) are independent of \( i \).

Proof. It is sufficient to consider \((a, u_i, b) \). Let \( n(a, b) = \max \{n(a), n(b)\} \) and choose \( i \) and \( j \geq n(a, b) \). It follows that

\[
((a, u_i, b), (a, u_j, b)) \Rightarrow ((a, u_i, b), (u_i, a, v_i), (a, u_j, b)) \\
= (a, u_i, (b, v_i, (a, u_j, b))) \\
= (a, u_i, (b, a, (v_i, u_j, b))) \\
= (a, u_i, (b, a, b)) = (a, u_i, b).
\]

Similarly \((a, u_i, b), (a, u_j, b) = (a, u_j, b)\).

(5.2) Postulates M, N, and O characterize distributive lattices.

Proof. Assume (M, N, O). Choose \( i = n(a, b) \) as in (5.1) and define \( a + b = (a, u_i, b) \) and \( ab = (a, v_i, b) \). As in [4, §12], \( S \) is a distributive lattice if \( a(a + b) = a \). Let \( j = n(a, a + b) \). Then \( a(a + b) = (a, v_j, (a, u_i, b)) \)
Let \((a, b, (a, u_i, v_j)) = (a, b, a) = a\). Conversely, in a distributive lattice \(S\) we may introduce medians with properties M and N \([4, \S 10]\). Let \(a_1, a_2, \ldots\) be a well-ordering of elements of \(S\). If we define \(u_i = \prod_{n=1}^{i} a_n\) and \(v_i = \sum_{n=1}^{i} a_n\), then clearly property O follows.

6. **Remarks on medians.** Defining lattice betweenness as in \([3]\), a well-known five-element lattice \([1, \text{Chap. V, Theorem 2}]\) shows that in a lattice medians need not exist. Another well-known five-element lattice \([1, \text{Chap. IX, Theorem 2}]\) shows that, even in a modular lattice, medians need not be unique. It is to be expected that medians (as defined here), to the extent they are associated with lattices, are associated with distributive lattices. See \([5]\).

Another type of "median," useful in showing independence of certain properties, seems of some intrinsic interest. Consider points in the Euclidean plane. Let \((x, y)\), the "segment" from \(x\) to \(y\), consist of all points in the closed convex lens bounded by circular arcs \(xy\), of radius \(d/3^{1/2}\) where \(d\) is the distance from \(x\) to \(y\). If \(r\) lies in this segment, we have the betweenness relation \(xry\). It follows that medians exist and are unique. The median \((x, y, z)\) proves to be the point associated with the solution of Steiner's celebrated minimal problem which deals with shortest highway networks serving towns \(x, y,\) and \(z\) \([2]\).

**Bibliography**