MEDIANS AND BETWEENNESS[1]

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In a recent paper [**4**][2] it is shown that lattices and trees have a common generalization. These systems will be called median semilattices. These semilattices are characterized below by means of segments (§§2 and 4) and by means of betweenness (§§3 and 4). In §5 distributive lattices are characterized by means of medians with no assumption made as to the existence of universal bounds (compare Problem 66 of [**1**]). In another paper [**5**], median semilattices are characterized in terms of a binary operation and it is shown that these systems can be imbedded in distributive lattices.

1. Medians. Consider a set S closed under a ternary operation (a, b, c), called the median of a, b, and c, satisfying the identities:

(M) $(a, a, b) = a$.

(N) $((a, b, c), (a, b, d), e) = ((c, d, e), a, b)$.

We call S a median semilattice.

We use freely properties previously derived for the median [**4**, §8]. In particular we note (a, b, c) is invariant under permutations of a, b, and c. We say x is between a and b and write axb if and only if $x = (a, x, b)$. The segment (a, b) is defined as the set of elements between a and b. It is immediate that $(a, a) = \{a\}$ and that $(a, b) = (b, a)$ contains a and b.

Theorems 1.1, 1.2, and 1.3 show that Postulates M and N imply Postulates P, Q, and R of the next section.

(1.1) $(a, b) \cap (a, c) = (a, d)$ *where* $d = (a, b, c)$.

Proof. Let $x = (a, x, b) = (a, x, c)$. Then $x = (x, x, a) = ((a, x, b), (a, x, c), a) = ((b, c, a), a, x) = (a, x, d)$. Conversely, let $x = (a, x, d)$. From [**4**, (8.4) and (8.6)], $d = (a, d, b)$ and $x = (a, x, (a, d, b)) = (a, b, (a, d, x)) = (a, b, x)$. Similarly, $x = (a, x, c)$.

(1.2) $(a, b) \subset (a, c)$ *implies for all* x *that* $(x, a) \cap (x, c) \subset (x, b)$.

Proof. We note $b = (a, b, c)$. Let $r = (a, r, x) = (c, r, x)$. Then $r = (r, r, b) = ((x, r, a), (x, r, c), b) = ((a, c, b), x, r) = (b, x, r)$.

(1.3) $(a, b) \cap (a, c) = \{a\}$ *implies* $a \in (b, c)$.

Proof. It is given that $r = (a, r, b) = (a, r, c)$ implies $r = a$. It follows, from [**4**, (8.4)], that $(b, a, c) = a$.

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

Theorems 1.4 and 1.5 are for future reference. The latter theorem is of some intrinsic interest.

(1.4) $abc\cdot adc\cdot bxd \rightarrow cxa$.

PROOF. Since $b=(a, b, c)$ and $d=(a, d, c)$, $x=(b, x, d)=((a, b, c), x, (a, d, c))=(a, c, (b, x, d))=(a, c, x)$.

(1.5) $(a, x, b)=(a, y, b)$ *if and only if* $(x, a, y)=(x, b, y)$.

PROOF. By symmetry, it is sufficient to prove necessity. Let $z=(x, b, y)$ where $(a, x, b)=(a, y, b)$. Then $(x, a, y)=((y, b, y), a, x)=((a, x, y), (a, x, b), y)=((a, x, y), (a, y, b), y)=(z, a, y)$. Similarly $(x, a, y)=(z, a, x)$. Hence $z=(z, z, a)=((x, y, b), z, a)=((z, a, x), (z, a, y), b)=(x, a, y)$.

2. **Median segments.** Consider a set S of elements $a, b, c, \cdots$. The symbol (a, b), called the segment from a to b, represents a subset of S determined by the ordered pair a, b. These segments have as properties:

(P) Given a, b, and c, there exists an element d such that $(a, b)\cap(a, c)=(a, d)$.

(Q) $(a, b)\cap(a, c)=(a, b)$ implies $(x, a)\cap(x, c)\subset(b, x)$ for all x.

(R) $(a, b)\cap(a, c)=(a, a)$ implies $(a, a)\cap(b, c)=\{a\}$.

We say b is between a and c and write abc if and only if $b\in(a, c)$. Theorems 2.2, 2.8, and Postulate Q show that S satisfies Postulates B_1, D_1, and I of the next section.

The following result is obtained from (Q) by choosing $a=b=c$.

(2.1) $(a, b)=(b, a)$.

From (R), with $a=b=c$, we have

(2.2) $(a, a)=\{a\}$.

(2.3) *Both a and b belong to* (a, b).

PROOF. Let $(a, b)\cap(a, a)=(a, d)$. Since $(a, d)\subset(a, a)$, we have from (Q) that $(a, a)\cap(a, a)\subset(a, d)$ and hence $a\in(a, a)\subset(a, d)\subset(a, b)$. That $b\in(a, b)$ now follows from (2.1).

(2.4) $(a, b)\cap(a, c)\supset(a, d)$ *implies that* $(a, d)\cap(b, c)\subset(d, d)$.

PROOF. We apply (Q) twice. From $(a, d)\subset(a, b)$ we have $(c, a)\cap(c, b)\subset(c, d)$. From $(a, d)\subset(a, c)$, $(d, a)\cap(d, c)\subset(d, d)$. Thus $(a, d)\cap(b, c)=(a, d)\cap(a, c)\cap(b, c)\subset(a, d)\cap(c, d)\subset(d, d)$.

(2.5) *The relations* $b\in(a, c)$, $(a, b)\subset(a, c)$, *and* $(a, b)\cap(b, c)=(b, b)$ *are equivalent.*

PROOF. Assume $b\in(a, c)$ and let $(a, b)\cap(a, c)=(a, d)$. Then from (2.3) and (2.4), $b\in(a, b)\cap(a, c)\cap(b, c)\subset(d, d)$. Hence $b=d$ and $(a, b)=(a, d)\subset(a, c)$. Next, if $(a, b)\subset(a, c)$, $(b, a)\cap(b, c)\subset(b, b)$. Since $b\in(b, a)\cap(b, c)$, $(b, a)\cap(b, c)=\{b\}$. Finally if $(b, a)\cap(b, c)=(b, b)$, we have from (R) that $b\in(a, c)$.

We have as a corollary

(2.6) $(a, b)=(a, c)$ *implies* $b=c$.

(2.7) *If* $(a, b)\cap(a, c)=(a, d)$, *then* $(a, d)\cap(b, c)=\{d\}$.

PROOF. From $d\in(a, d)$, we have $d\in(b, a)$ and $d\in(c, a)$. Hence $(b, d)\subset(b, a)$, $(c, d)\subset(c, a)$, and it follows that $(b, d)\cap(c, d)\subset(a, d)$. Moreover, we have $(b, d)\cap(d, a)=(d, d)=(c, d)\cap(d, a)$. Hence $(d, d)=(a, d)\cap(b, d)\cap(c, d)=(b, d)\cap(c, d)$, and $d\in(b, c)$. This, with (2.4), proves the theorem.

The element d, whose existence is given by (P) and whose uniqueness is implied by (2.6), is called the median of a, b, c and is denoted by (a, b, c). As a corollary of (2.7) we have

(2.8) $(a, b)\cap(b, c)\cap(c, a)=\{d\}$ *where* $d=(a, b, c)$.

From (2.1) and (2.8) it is clear that (a, b, c) is invariant under permutations of a, b, and c.

3. **Median betweenness.** Consider a set S in which there exists a ternary relation called betweenness. If b is between a and c, we write abc. This relation has the properties:

(D_1) To a, b, and c corresponds an element w such that $awb\cdot bwc\cdot cwa$.

(B_1) $aba\rightarrow a=b$.

(I) $abc\cdot abd\cdot cxd\rightarrow xba$.

Setting $a=c$ in (D_1) we have from (B_1) that

(3.1) *Both aab and baa hold.*

Setting $c=x=d$ in (I), we have

(3.2) $abc\rightarrow cba$.

Setting $a=x$ and $b=d$ in (I), we have

(3.3) $abc\cdot bac\rightarrow a=b$.

Since (I) is Transitivity T_7 of [3], it follows from that paper (p. 102) that

(3.4) $abc\cdot acd\rightarrow bcd$.

(3.5) *Element* w *in* (D_1) *is unique.*

PROOF. Assume $axb\cdot bxc\cdot cxa$. By (I), $awb\cdot awc\cdot bxc\rightarrow xwa$ and $axb\cdot axc\cdot bwc\rightarrow wxa$. By (3.3), $x=w$.

We call element w in (D_1) the median of a, b, and c and write $w=(a, b, c)$. By (3.2), (a, b, c) is invariant under permutations.

(3.6) $abc\cdot acd\rightarrow abd$.

PROOF. Let $r=(a, b, d)$. We have $bra\cdot brd\cdot acd\rightarrow crb$, $crb\cdot cba\rightarrow rba$, and $bra\cdot rba\rightarrow b=r$. Thus ard implies abd.

(3.7) $abc\cdot bcd\cdot ade\rightarrow ecb$.

PROOF. Let $r=(e, c, b)$. We have $crb\cdot cba\rightarrow cra$, $cra\cdot cre\cdot ade\rightarrow drc$, $brc\cdot bcd\rightarrow rcd$, $crd\cdot rcd\rightarrow r=c$, and $erb\rightarrow ecb$.

(3.8) *Postulates* M *and* N *hold.*

PROOF. (M) follows from (3.1) and (3.5). In proving (N) we let

$r=(a, b, c)$, $s=(a, b, d)$, $p=(r, s, e)$, $t=(c, d, e)$, and $q=(t, a, b)$. Using (I), (3.7), and (3.4) we have

$$\begin{aligned}
cra \cdot crb \cdot aqb &\rightarrow qrc,\\
dsa \cdot dsb \cdot aqb &\rightarrow qsd,\\
tqa \cdot tqb \cdot arb &\rightarrow rqt,\\
tqa \cdot tqb \cdot asb &\rightarrow sqt,\\
dsq \cdot sqt \cdot dtc &\rightarrow cqs,\\
crq \cdot cqs &\rightarrow rqs,\\
epr \cdot eps \cdot rqs &\rightarrow qpe,\\
tqr \cdot tqs \cdot rps &\rightarrow pqt,\\
epq \cdot pqt \cdot etc &\rightarrow cqp,\\
epq \cdot pqt \cdot etd &\rightarrow dqp,\\
crq \cdot cqp &\rightarrow rqp,\\
dsq \cdot dqp &\rightarrow sqp,\\
pqr \cdot pqs \cdot rps &\rightarrow pqp.
\end{aligned}$$

Hence $p=q$ and (N) is proved.

We may summarize the results of these sections by the following theorem.

(3.9) A median semilattice is characterized by (M, N), by (P, Q, R), or by (B_1, D_1, I).

4. **Further characterizations.** In this section we refer to some additional postulates:

(D) To a, b, and c there corresponds a unique element w such that $awb \cdot bwc \cdot cwa$.

(A) $abc \rightarrow cba$.

(B_2) $abc \cdot acb \leftrightarrow b=c$.

(E) $abc \cdot acd \rightarrow dcb$.

(F) $abc \cdot acd \rightarrow dba$.

(G) $abc \cdot adc \cdot bxd \rightarrow cxa$.

(H) $abc \cdot acd \cdot ade \rightarrow edb$.

(J) $abc \cdot bcd \cdot ade \rightarrow ecb$.

Postulates A through J are variants respectively of Postulates α, β, t_1, t_2, T_5, T_6, and T_4 of [**3**]. The following notation is used for sets of postulates: Σ_0 for (D, B_1, E), Σ_1 for (D, B_1, F), Σ_2 for (D, B_1, G), Σ_3 for (D, B_1, H), Σ_4 for (D_1, B_1, J), and Σ_5 for (D_1, B_1, I).

In the previous section we proved Σ_5 characterizes median semilattices. By that section and [**4**], Σ_5 implies Postulates D, E, F, and J.

By (1.4), Σ_5 implies (G). By [**3**], Σ_5 implies (H). We summarize as follows:

(4.1) Σ_5 *implies* Σ_i *for* $i=0, 1, 2, 3, 4$.

The following theorem holds since, by the previous section, the properties stated are consequences of (D_1) and (B_1).

(4.2) Σ_i *implies* $aab \cdot baa$ *for* $i=0, 1, 2, 3, 4$. *Hence if abc holds, element w of* (D_1) *may be chosen as b.*

(4.3) Σ_i *implies* (A) *for* $i=0, 1, 2, 3, 4$.

Proof. In (E), let $a=b$. In (F), let $b=c$. In (G), let $b=x=d$. In (H), let $a=b=c$. In (J), let $a=b$ and $c=d$.

(4.4) Σ_i *implies* (B_2) *for* $i=0, 1, 2, 3, 4$.

Proof. For $i=0, 1, 2, 3$, this is a consequence of (4.2) and (D). For Σ_4, choose $a=d$ and $b=e$ in (J).

(4.5) Σ_4 *implies* (I).

Proof. By (4.3), (4.4), and [**3**] we have (E). To prove (I), assume $abc \cdot abd \cdot cxd$. By (D_1), for some element r, $arb \cdot brx \cdot xra$. We have $arb \cdot abc \to rbc$, $cbr \cdot brx \cdot cxd \to drb$, $arb \cdot abd \to rbd$, and $dbr \cdot drb \to r=b$. Thus $xra \to xba$.

In discussing Σ_i, $i=0, 1, 2, 3$, we may denote element w of (D) by (a, b, c). By (4.3), (a, b, c) is invariant under permutations.

(4.6) Σ_2 *and* Σ_3 *each imply* Σ_1.

Proof. By (4.3), (4.4), and [**3**], (G) implies (F), and (H) implies (E). It remains to show (H) implies (F). Assume $abc \cdot acd$. Let $r=(a, b, d)$. Then $arb \cdot abc \to rbc$, $arb \cdot abc \cdot acd \to dcr$, $dcr \cdot drb \to crb$, and $crb \cdot cbr \to b=r$. Thus $ard \to abd$.

(4.7) Σ_1 *implies* (E).

Proof. Assume $abc \cdot acd$. Let $x=(b, c, d)$. Then $dxc \cdot dca \to dxa$, $cxb \cdot cba \to cxa$, and $x=(a, c, d)=c$. Thus $bxd \to bcd$.

As in [**4**] it is convenient to extend the betweenness notation to a notation for chains. Thus $abcd$ denotes $abc \cdot abd \cdot acd \cdot bcd$. If Σ_1 holds, by repeated use of (E) and (F) we may obtain implications such as $abcd \cdot bxc \to abxcd$.

(4.8) *If* Σ_1 *holds*, $amp \cdot amq \cdot pmq \cdot mpx \cdot mqx \cdot pxq \to apx$.

Proof. Let $r=(a, p, x)$, $s=(a, q, x)$, and $t=(a, r, s)$. Since $atr \cdot arp \to atp$, $ats \cdot asq \to atq$, and $pxq \cdot prx \cdot xsq \cdot rts \to prtsq$, we have $t=(a, p, q)=m$. It follows that $amr \cdot arp \to mrp$ and $mrp \cdot mpx \cdot prx \to mrprx \to r=p$. Finally $arx \to apx$.

(4.9) Σ_1 *implies*(I).

Proof. Given $abc \cdot abd \cdot cxd$. Let $m=(b, c, d)$, $p=(m, c, x)$, and $q=(m, d, x)$. Then $abc \cdot bmc \to amc$, $abd \cdot bmd \to amd$, and $m=(a, c, d)$. Thus $amc \cdot mpc \to amp$, and $amd \cdot mqd \to amq$. Also $cxd \cdot cpx \cdot xqd \to pxq$ and $cmd \cdot cpm \cdot mqd \to pmq$. Since mpx and mqx hold, we have by (4.8) that apx holds. Finally $abc\cdot bmc\cdot mpc\rightarrow abmpc$ and $abp\cdot apx\rightarrow abx$.

As a consequence of (4.1), (4.5), (4.6), and (4.9), we have the following theorem.

(4.10) *Median semilattices are characterized by* Σ_i *for* $i=1, 2, 3, 4, 5$.

A particularly simple characterization in terms of segments may be based on Σ_1. Postulates used are:

(α) Given a, b, and c, there exists w such that $(a, b)\cap(b, c)\cap(c, a)=\{w\}$.

(β) $(a, a)=\{a\}$.

(γ) If $b\in(a, c)$, then $(a, b)\subset(a, c)\cap(c, a)$.

(4.11) *Median semilattices are characterized by* (α, β, γ).

Proof. Let $c=a$ in (α). Since $w\in(a, a)=\{a\}$, $w=a$. It follows that $a\in(a, b)\cap(b, a)$. Since $a\in(b, a)$ and $b\in(a, b)$, we have from (γ) that $(a, b)=(b, a)$. Clearly $(a, b)\subset(a, c)$ implies $b\in(a, c)$. It is now easy to prove (F) in the form: if $b\in(a, c)$ and $c\in(a, d)$, then $b\in(a, d)$.

The five-element modular nondistributive lattice (see §6) shows that, in Σ_3, we cannot weaken (D) to (D_1) and still obtain a median semilattice. As an example showing that we cannot weaken (D) in Σ_1 or in Σ_2 we have $\{a, b, c, r, s\}$ where $ars\cdot brs\cdot crs$ and where both r and s are "medians" of a, b, and c. It has not been determined whether Σ_0 characterizes median semilattices or not.

5. **Distributive lattices.** Consider a median semilattice S which satisfies the following postulate:

(O) S contains sequences $u_1, u_2, \cdots$ and $v_1, v_2, \cdots$ such that to each a in S there corresponds an ordinal $n(a)$ with the property $(u_i, a, v_j)=a$ for $i, j\geqq n(a)$.

(5.1) *To each pair of elements a and b there corresponds an ordinal* $n(a, b)$ *such that for* $i\geqq n(a, b)$ *both* (a, u_i, b) *and* (a, v_i, b) *are independent of* i.

Proof. It is sufficient to consider (a, u_i, b). Let $n(a, b)=\max[n(a), n(b)]$ and choose i and $j\geqq n(a, b)$. It follows that

$$\begin{aligned}((a, u_i, b), a, (a, u_j, b)) &= ((a, u_i, b), (u_i, a, v_i), (a, u_j, b))\\ &= (a, u_i, (b, v_i, (a, u_j, b)))\\ &= (a, u_i, (b, a, (v_i, u_j, b)))\\ &= (a, u_i, (b, a, b)) = (a, u_i, b).\end{aligned}$$

Similarly $((a, u_i, b), a, (a, u_j, b))=(a, u_j, b)$.

(5.2) *Postulates* M, N, *and* O *characterize distributive lattices.*

Proof. Assume (M, N, O). Choose $i=n(a, b)$ as in (5.1) and define $a+b=(a, u_i, b)$ and $ab=(a, v_i, b)$. As in [**4**, §12], S is a distributive lattice if $a(a+b)=a$. Let $j=n(a, a+b)$. Then $a(a+b)=(a, v_j, (a, u_i, b))$

$=(a, b, (a, u_i, v_j))=(a, b, a)=a$. Conversely, in a distributive lattice S we may introduce medians with properties M and N [**4**, §10]. Let $a_1, a_2, \cdots$ be a well-ordering of elements of S. If we define $u_i=\prod_{n=1}^{i} a_n$ and $v_i=\sum_{n=1}^{i} a_n$, then clearly property O follows.

6. **Remarks on medians.** Defining lattice betweenness as in [**3**], a well-known five-element lattice [**1**, Chap. V, Theorem 2] shows that in a lattice medians need not exist. Another well-known five-element lattice [**1**, Chap. IX, Theorem 2] shows that, even in a modular lattice, medians need not be unique. It is to be expected that medians (as defined here), to the extent they are associated with lattices, are associated with distributive lattices. See [**5**].

Another type of "median," useful in showing independence of certain properties, seems of some intrinsic interest. Consider points in the Euclidean plane. Let (x, y), the "segment" from x to y, consist of all points in the closed convex lens bounded by circular arcs xy, of radius $d/3^{1/2}$ where d is the distance from x to y. If r lies in this segment, we have the betweenness relation xry. It follows that medians exist and are unique. The median (x, y, z) proves to be the point associated with the solution of Steiner's celebrated minimal problem which deals with shortest highway networks serving towns x, y, and z [**2**].

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