

A PROPERTY OF EXTREME POINTS OF COMPACT CONVEX SETS

MEYER JERISON

1. **Introduction.** The knowledge that a given convex set has extreme points can be very useful, and this accounts for the fame of the Krein-Milman theorem. Sometimes, more detailed information about the location of the extreme points of the set is needed. A method for obtaining such information, first used by Milman in [3], is applied here to locate the extreme points of the limit of a sequence of convex sets.

We shall be dealing with a real, locally convex, linear Hausdorff space, E . If S is a subset of E , the closed convex hull of S will be denoted by S^\blacktriangle . For any f in E^* (the set of continuous linear functionals on E), we let $f_S = \sup_{x \in S} f(x)$. The properties of convex sets that are relevant to our problem may be summarized in one concise statement (supplied by the referee) as follows:

THEOREM 1. *If C is a compact convex subset of E , and $S \subset C$, then the following assertions are equivalent:*

- (i) $f_S = f_C$ for each $f \in E^*$;
- (ii) $C = S^\blacktriangle$;
- (iii) the closure of S includes all extreme points of C .

All of this is proved in [1, especially p. 84]. For instance, the assertion (iii) implies (ii) is the Krein-Milman theorem.

The technique that Milman uses in [3] is to prove (i) in order to infer (iii). Our application of it yields:

THEOREM 2. *Let $\{K_n\}$ be a sequence of compact convex sets whose union is contained in a compact convex subset of E ; let A_n be the set of extreme points of K_n for each n ; and let $K = \limsup K_n$ and $A = \limsup A_n$. Then $K \subset A^\blacktriangle$. If, in addition, K is convex, then $K = A^\blacktriangle$, that is, A contains all of the extreme points of K .*

The limit superior is used here in the topological sense, i.e., $x \in \limsup A_n$ if, and only if, every neighborhood of x meets sets of the sequence $\{A_n\}$ with arbitrarily large index n . Observe that it is always a closed set.

To the best of my knowledge, Theorem 2 is new even if E is finite-dimensional. It has been applied in [2] to help locate the extreme

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points of the set of generalized limits of bounded sequences.

2. Proof of Theorem 2. We treat first the case of a decreasing sequence $\{K_n\}$. Under that hypothesis, $K = \bigcap_{n=1}^{\infty} K_n$, and since it is convex, we should obtain the stronger conclusion $K = A^\blacktriangle$.

Let F_n be the *closure* of $\bigcup_{m=n}^{\infty} A_m$, so that $A = \bigcap_{n=1}^{\infty} F_n$. Since $F_n \subset K_n$ for each n , $A \subset K$, and therefore, for each $f \in E^*$, $f_A \leq f_K$. The desired conclusion will follow from Theorem 1 if we succeed in reversing this inequality. By definition, F_n contains all of the extreme points of K_n , and this implies that $f_{F_n} = f_{K_n}$. Since F_n is compact, it contains a point x_n such that $f(x_n) = f_{F_n}$. The sequence $\{x_n\}$ has a cluster point x_0 which is necessarily in A , and

$$f_A \geq f(x_0) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} f_{K_n} \geq f_K.$$

(The limit exists because f_{K_n} is a monotone sequence bounded by f_K .)

The general case is handled by applying the result already obtained to the decreasing sequence $\{F_n^\blacktriangle\}$ of compact convex sets. Since all of the extreme points of F_n^\blacktriangle are in the closed set F_n (Theorem 1 (iii)), we have $\limsup (F_n^\blacktriangle) = (\limsup F_n)^\blacktriangle = (\bigcap_{n=1}^{\infty} F_n)^\blacktriangle = A^\blacktriangle$. But $K_n = A_n^\blacktriangle \subset F_n^\blacktriangle$, and therefore,

$$K = \limsup K_n \subset \limsup F_n^\blacktriangle = A^\blacktriangle.$$

Since $A \subset K$, $A^\blacktriangle \subset K$ if K is convex, which completes the proof of the theorem.

The theorem has been so formulated as to include the case of an increasing sequence $\{K_n\}$, where we conclude that A^\blacktriangle is the closure of $\bigcup_{n=1}^{\infty} K_n$. Also, in the general case, $\bigcap_{n=1}^{\infty} K_n \subset A^\blacktriangle$. It may be observed, in conclusion, that the sequence of sets can be replaced by a directed system of sets.

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PURDUE UNIVERSITY