A PROPERTY OF EXTREME POINTS OF COMPACT CONVEX SETS

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1. **Introduction.** The knowledge that a given convex set has extreme points can be very useful, and this accounts for the fame of the Krein-Milman theorem. Sometimes, more detailed information about the location of the extreme points of the set is needed. A method for obtaining such information, first used by Milman in [**3**], is applied here to locate the extreme points of the limit of a sequence of convex sets.

We shall be dealing with a real, locally convex, linear Hausdorff space, E. If S is a subset of E, the closed convex hull of S will be denoted by $S^{\blacktriangle}$. For any f in E^* (the set of continuous linear functionals on E), we let $f_S = \sup_{x \in S} f(x)$. The properties of convex sets that are relevant to our problem may be summarized in one concise statement (supplied by the referee) as follows:

THEOREM 1. *If C is a compact convex subset of E, and $S \subset C$, then the following assertions are equivalent*:

(i) $f_S = f_C$ *for each* $f \in E^*$;

(ii) $C = S^{\blacktriangle}$;

(iii) *the closure of S includes all extreme points of C.*

All of this is proved in [**1**, especially p. 84]. For instance, the assertion (iii) implies (ii) is the Krein-Milman theorem.

The technique that Milman uses in [**3**] is to prove (i) in order to infer (iii). Our application of it yields:

THEOREM 2. *Let $\{K_n\}$ be a sequence of compact convex sets whose union is contained in a compact convex subset of E; let A_n be the set of extreme points of K_n for each n; and let $K = \lim\sup K_n$ and $A = \lim\sup A_n$. Then $K \subset A^{\blacktriangle}$. If, in addition, K is convex, then $K = A^{\blacktriangle}$, that is, A contains all of the extreme points of K.*

The limit superior is used here in the topological sense, i.e., $x \in \lim\sup A_n$ if, and only if, every *neighborhood* of x meets sets of the sequence $\{A_n\}$ with arbitrarily large index n. Observe that it is always a closed set.

To the best of my knowledge, Theorem 2 is new even if E is finite-dimensional. It has been applied in [**2**] to help locate the extreme

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points of the set of generalized limits of bounded sequences.

2. **Proof of Theorem 2.** We treat first the case of a decreasing sequence $\{K_n\}$. Under that hypothesis, $K=\bigcap_{n=1}^{\infty}K_n$, and since it is convex, we should obtain the stronger conclusion $K=A^{\blacktriangle}$.

Let F_n be the *closure* of $\bigcup_{m=n}^{\infty}A_m$, so that $A=\bigcap_{n=1}^{\infty}F_n$. Since $F_n\subset K_n$ for each n, $A\subset K$, and therefore, for each $f\in E^*$, $f_A\leqq f_K$. The desired conclusion will follow from Theorem 1 if we succeed in reversing this inequality. By definition, F_n contains all of the extreme points of K_n, and this implies that $f_{F_n}=f_{K_n}$. Since F_n is compact, it contains a point x_n such that $f(x_n)=f_{F_n}$. The sequence $\{x_n\}$ has a cluster point x_0 which is necessarily in A, and

$$f_A \geqq f(x_0) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f_{K_n} \geqq f_K.$$

(The limit exists because f_{K_n} is a monotone sequence bounded by f_K.)

The general case is handled by applying the result already obtained to the decreasing sequence $\{F_n^{\blacktriangle}\}$ of compact convex sets. Since all of the extreme points of $F_n^{\blacktriangle}$ are in the closed set F_n (Theorem 1 (iii)), we have $\limsup (F_n^{\blacktriangle})=(\limsup F_n)^{\blacktriangle}=(\bigcap_{n=1}^{\infty}F_n)^{\blacktriangle}=A^{\blacktriangle}$. But $K_n=A_n^{\blacktriangle}\subset F_n^{\blacktriangle}$, and therefore,

$$K = \limsup K_n \subset \limsup F_n^{\blacktriangle} = A^{\blacktriangle}.$$

Since $A\subset K$, $A^{\blacktriangle}\subset K$ if K is convex, which completes the proof of the theorem.

The theorem has been so formulated as to include the case of an increasing sequence $\{K_n\}$, where we conclude that $A^{\blacktriangle}$ is the closure of $\bigcup_{n=1}^{\infty}K_n$. Also, in the general case, $\bigcap_{n=1}^{\infty}K_n\subset A^{\blacktriangle}$. It may be observed, in conclusion, that the sequence of sets can be replaced by a directed system of sets.

References

1. N. Bourbaki, *Espaces vectoriels topologiques*, Actualités Scientifiques et Industrielles, no. 1189, Paris, 1953.

2. M. Jerison, *The set of all generalized limits of bounded sequences*, to be published.

3. D. Mil'man, *Isometry and extremal points*, Doklady Akad. Nauk SSSR (N.S.) vol. 59 (1948) pp. 1241–1244.

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