CONTRACTIBILITY AND CONVEXITY[1]

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The purpose of this note is to present a purely geometrical result that appeared as a by-product of research in the theory of games. A preliminary draft had been written when it was discovered that a proof of the theorem had been published by I. Liberman [1][2] in 1943. However, the present proof seems to retain its interest, both for its conciseness and the fact that it exposes a connection between the geometry of convex sets and fixed-point theory that was motivated by game theory.

The result in question is:

THEOREM. *In order that a locally contractible compact set X in n-dimensional Euclidean space be convex, it is necessary and sufficient that the set X be contractible and that every support contact of X be contractible.*

By a *support contact* of X is meant the intersection of X with a supporting hyperplane, that is, the set of $x=(x_1, \cdots, x_n)\in X$ for which a given linear form $(a, x)=a_1x_1+\cdots+a_nx_n$ assumes its maximum value α; support contacts of convex sets are obviously convex. The necessity of the conditions is clear since a convex set X can always be contracted to any point $x_0\in X$ by the contraction $h(x, t) =tx_0+(1-t)x$ where $x\in X$ and $0\leqq t\leqq 1$. To prove the sufficiency of the conditions, assume that X is not convex and let $C\neq X$ be the convex hull of X. Then there exists a point p on the boundary of C that does not lie in X (otherwise X would contain the boundary of C which is a topological sphere, but not some point inside the sphere, contradicting the contractibility of X). Since X is compact, C is also and there exists a hyperplane supporting C through p. Choose coordinates in the Euclidean space so that p is the origin O and the supporting hyperplane is given by $x_1+\cdots+x_n=0$, that is, $x_1+\cdots+x_n\leqq 0$ for all $x\in C$. Relative to this choice of coordinates, consider the zero-sum two-person game played thus:

Move 1. Player I chooses a point $x\in X$.
Move 2. Player II chooses an integer $i=1, \cdots, n$.
Payoff. Player II pays Player I the amount x_i.

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

If Player II plays the integer i with probability y_i, where $y_i \geqq 0$ and $y_1 + \cdots + y_n = 1$, then the expected payoff is

$$f(x, y) = x_1 y_1 + \cdots + x_n y_n. \tag{1}$$

Since $x_1 + \cdots + x_n \leqq 0$ for all $x \in X$ and $0 \notin X$, it is clear that every $x \in X$ has at least one negative coordinate and hence

$$\max_x \min_y f(x, y) = \max_x \min_i x_i < 0, \tag{2}$$

where the existence of the maximum is assured by the compactness of X. Thus the value of the game to Player I is negative. On the other hand, suppose Player II can choose probabilities $\bar{y}_1, \cdots, \bar{y}_n$ such that $\max_x f(x, \bar{y}) = \alpha < 0$. Then $(\bar{y}, x) = \alpha$ is a supporting hyperplane for X and hence for C. However $0 \in C$ and $(\bar{y}, 0) = 0 > \alpha$. Hence no such $\bar{y}$ exists and

$$\min_y \max_x f(x, y) \geqq 0. \tag{3}$$

Thus the value of the game to Player II is nonpositive, and the game does not have a value.

In a recent paper [**2**], Debreu has observed that fixed-point theorems of Eilenberg and Montgomery [**3**] or Begle [**4**] imply the following theorem on zero-sum two-person games:

Let X and Y be contractible and locally contractible compact sets in n-dimensional Euclidean space, and $f(x, y)$ a continuous real-valued function defined on $X \times Y$. If the sets

$$U_{\bar{x}} = \left\{ y \in Y \,\middle|\, f(\bar{x}, y) = \min_y f(\bar{x}, y) \right\},$$

$$V_{\bar{y}} = \left\{ x \in X \,\middle|\, f(x, \bar{y}) = \max_x f(x, \bar{y}) \right\}$$

are contractible for all $\bar{x} \in X$ and all $\bar{y} \in Y$, then

$$\max_x \min_y f(x, y) = \min_y \max_x f(x, y).$$

In the language of game theory, if X and Y are the sets of strategies and $f(x, y)$ the payoff function for a zero-sum two-person game, then this theorem asserts that the game has a value. For the game defined above, the sets $U_{\bar{x}}$ are faces of the simplex Y of probabilities $y_1, \cdots, y_n$ since the function f is linear. On the other hand, the sets $V_{\bar{y}}$ are support contacts of X and hence are contractible by assumption. Therefore the game has a value, contradicting conclusions (2) and (3) above. The assumption that X is not convex falls with them and this completes the proof of the theorem.

Liberman's theorem is stronger than the result just proved in one important respect; it does not presuppose local contractibility. This condition entered the proof by way of the fixed-point theorems and it does not seem possible to eliminate it directly without a stronger theorem than seems to exist in the literature. Indeed, it would require the following conjectural statement:

Let X be a contractible compact set in n-dimensional Euclidean space and T an upper semi-continuous mapping which assigns to each point $x \in X$ an acyclic subset $T(x) \subset X$. Then T has a fixed point, i.e., for some x, $T(x)$ contains x.

Although the special case of a single-valued mapping is a well-known and long-standing problem (an example of Borsuk [**5**] shows that it is not enough to assume that X is an acyclic compact set), the validity of Liberman's theorem seems to indicate possible progress in this direction.

(Added March 30, 1954. The discovery by S. Kinoshita of a contractible continuum in 3-dimensional Euclidean space without fixed point property (Fund. Math. vol. 40 (1953) pp. 96–98) means that Liberman's theorem cannot be proved directly by means of a fixed point theorem.)

Bibliography

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