REMARKS ON PRIMITIVE IDEMPOTENTS IN
COMPACT SEMIGROUPS WITH ZERO

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We recall that a mob is a Hausdorff space together with a continuous associative multiplication. A nonempty subset $A$ of a mob $X$ is a submob if $AA \subseteq A$. This note consists of an amplification of results of Numakura dealing with primitive idempotents in a compact mob $X$ with zero (see definitions below). We discuss the properties of certain “fundamental” sets determined by primitive idempotents, namely the sets $XeX$, $Xe$, $eX$, and $eXe$, where $e$ is a primitive idempotent. These are, respectively, the smallest (two-sided, left, right, bi-) ideal containing $e$. Included in Theorem 1 is a characterization of a primitive idempotent in terms of its “fundamental” sets. There then follow some remarks on the structure of the smallest ideal containing the set of all primitive idempotents.

Finally, if $e$ is a nonzero primitive idempotent of the compact connected mob $X$ with zero, then the set of nilpotent elements of $X$ is dense in each of the “fundamental” sets determined by $e$.

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We shall assume throughout most of this note that $X$ is a compact mob with zero (0). For $a \in X$ we denote by $\Gamma(a)$ the closure of the set of positive powers of $a$, and by $K(a)$ the minimal (closed) ideal of $\Gamma(a)$. $K(a)$ is known to be a (topological) group and consists of the cluster points of the set of powers of $a$ ([3; 5]; these results depend only on the compactness of $\Gamma(a)$). Also $\Gamma(a)$ contains exactly one idempotent, $e$, and if $e = 0$ then the powers of $a$ converge to 0. An element $a$ is termed nilpotent if its powers converge to 0, and we denote by $N$ the set of all nilpotent elements of $X$. A subset $A$ of $X$ is termed nil if $A \subseteq N$. An idempotent $e$ of $X$ is primitive if $g = g^{2} \in eXe$ implies $g = 0$ or $g = e$. Recall that a subset $A$ of $X$ is a bi-ideal if

1. $AA \subseteq A$
2. $AXA \subseteq A$ [2; 3].

Lemma 1. Let $e$ be an idempotent of the compact mob $X$ and denote by $\mathcal{M}(e)$ the collection of sets $Xf_{a}$, where $f_{a}$ is an idempotent of $XeX$. Let $\mathcal{M}(e)$ be partially ordered by inclusion; then $Xe$ is a maximal member of $\mathcal{M}(e)$.

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PROOF. Suppose $f$ is an idempotent in $XeX$ and $Xe \subseteq Xf$. Then there are elements $a, b$ of $X$ so that $f = ab$ and we may assume $ae = a, eb = b$. Now $Xe \subseteq Xf$ implies $e \in Xf$, $ef = e$; hence for each positive integer $n$, $a^neb^n = a^{n-1}eab^{n-1} = a^{n-1}eb^{n-1} = \cdots = aeb = f$. Hence there is an idempotent $g \in \Gamma(a)$ and an element $h \in \Gamma(b)$ so that $f = geh$ [5]. We note $ge = g$, hence $f = gf = (ge)f = g(ef) = ge = g$ and $f = g = ge = fe$. Hence $f \in Xe$ and $Xf \subseteq Xe$, completing the proof.

We remark that the arguments used in the proof of the lemma are due to Rees [7] and Numakura [5].

COROLLARY. Let $e$ be an idempotent of the compact mob $X$; then $eXe$ is maximal among the sets $fXf$ where $P = f^X \subseteq XeX$.

PROOF. Suppose $f \in XeX$ and $eXe \subseteq fXf$. Then $ef = e = fe$ and $e \in Xf$; hence $Xe \subseteq Xf$ so $Xe = Xf$ by the theorem. Hence $f = fe = e$ and $eXe = fXf$.

**Lemma 2.** Let $M$ be a left (right, two-sided, bi-) ideal of a mob $X$ and suppose $a \in M$ with $\Gamma(a)$ compact; then $\Gamma(a) \subseteq M$.

PROOF. We give the proof for a bi-ideal $M$; the other proofs are similar. Since $M$ is a mob, the set of powers of $a$ is contained in $M$. Now $aK(a)a \subseteq MXM \subseteq M$ and $aK(a)a \subseteq K(a)$ since $K(a)$ is an ideal in $\Gamma(a)$. Hence $K(a) \cap M \neq \emptyset$, so that $K(a) \subseteq M$ since no group can properly contain a bi-ideal. It follows that $\Gamma(a) \subseteq M$.

**Lemma 3.** Let $X$ be a mob with zero and suppose $a \in X$ with $\Gamma(a)$ locally compact; then $a \in N$ implies $\Gamma(a) \cap N = \emptyset$.

PROOF. Suppose $x \in \Gamma(a) \cap N$; then $\{x^n\}$ converges to 0, so $0 \in \Gamma(x) \subseteq \Gamma(a)$. Hence $\Gamma(a)$ has a minimal ideal and we have from [3] that $\Gamma(a)$ is compact. Since $\Gamma(a) \cap N \neq \emptyset$, it follows that $K(a) \cap N \neq \emptyset$, hence $K(a) = 0$ and $a \in N$, a contradiction.

THEOREM 1. Let $e$ be a nonzero idempotent of the compact mob $X$ with zero; then these are equivalent:

1. $e$ is primitive.
2. $(eXe) \setminus N$ is a group.
3. $eXe$ is a minimal non-nil bi-ideal.
4. $Xe$ is a minimal non-nil left ideal.
5. $XeX$ is a minimal non-nil ideal.
6. Each idempotent of $XeX$ is primitive.

PROOF. (1) implies (2). We first show $(eXe) \setminus N$ is a mob. Since $e$ is assumed primitive, $(eXe) \setminus N$ has a unit $e$ and no other idempotents. Suppose $a, b \in (eXe) \setminus N$ and $ab \in N$; we claim $Xa = Xb = Xe$. Accord-
ing to [3] there is an idempotent \( f \in \Gamma(a) \) such that \( \cap_n Xa^n = Xf \). Now \( a \in N \) implies (Lemma 3) \( \Gamma(a) \cap N = \emptyset \), hence \( e \in \Gamma(a) \) and \( e = f \). Therefore \( \cap_n Xa^n = Xe \), so \( Xa \supset Xe \). Since \( a \in eXe \subseteq Xe \), \( Xa \subseteq Xe \) and the claim is established for \( a \). Similar arguments establish the claim for \( b \). Now using the claim and the fact that \( e \) is a unit for \( a \) and \( b \) one may verify that \( X(ab)^n = Xe \) for each positive integer \( n \). Hence \( Xe = \cap_n X(ab)^n = Xf \) for some idempotent \( f \) in \( \Gamma(ab) \) (see [3]); but \( ab \in N \) implies \( f = 0 \), \( Xe = 0 \), and \( e = 0 \), a contradiction. This shows \( (eXe) \setminus N \) we conclude as above that \( e \in \Gamma(y) \); since \( e \) is a unit for \( y \) it follows that \( K(y) = \Gamma(y) \) is a group [3] contained in \( (eXe) \setminus N \) by Lemmas 3 and 2. Hence \( y \) has an inverse in \( (eXe) \setminus N \), completing the proof of (2).

(2) implies (3). Let \( M \) be a non-nil bi-ideal of \( X \) contained in \( eXe \) and choose \( a \in M \setminus N \); then \( aXa \subseteq M \subseteq eXe \). Let \( f \) be a nonzero idempotent in \( aXa \); then since \( (eXe) \setminus N \) is a group and \( f \in N \), \( f = e \in M \). Hence \( eXe \subseteq MXe \subseteq M \).

(3) implies (4). Let \( P \) be a non-nil left ideal of \( X \) contained in \( Xe \) and choose \( a \in P \setminus N \). Then there is a nonzero idempotent \( f \in \Gamma(a) \), and \( Xf \subseteq P \). Hence \( eXf \subseteq eP \subseteq eXe \). Now since \( f \in Xe \), \( f = fe \) and \( (ef)(ef) = e(ef)f = eff = ef \) so that \( ef \) is idempotent. Note that \( ef \in N \), for otherwise \( ef = 0 \) and \( f = (fe)f = f(ef) = 0 \). Therefore \( eXf \) is a non-nil bi-ideal and hence coincides with \( eXe \). Since \( f \in P \), \( eXe = eXf \subseteq P \) and we conclude \( e \in P \), \( Xe \subseteq P \).

(4) implies (5). Let \( M \) be a non-nil ideal of \( X \) contained in \( Xe \), and let \( f \) be a nonzero idempotent in \( M \). Then there are elements \( a, b, c \) of \( X \) so that \( f = aeb \). Let \( g = bae \); then \( g^2 = baebae = bfae \) and \( g^3 = g^2 \). Note that \( bf \not= 0 \), since otherwise \( f = aeb = aebf = 0 \). Also \( g^2 = bf = bfaebf = bf \); hence \( g^2 = 0 \), otherwise \( bf = 0 \). Now \( g^2 \subseteq XfX \) and \( g^2 \subseteq Xe \), so by (4), \( Xe = Xg^2 \subseteq XfX \) and we conclude \( e \subseteq XfX \), \( XeX \subseteq M \).

(5) implies (6). Let \( f \) be a nonzero idempotent of \( XeX \) and suppose \( g \) is a nonzero idempotent with \( g \in XfX \) (hence \( gXg \subseteq XfX \)). Since \( f, g \in XeX \) we have \( XgX = XfX = XeX \) and \( f \in XgX \). It follows from the corollary to Lemma 1, then, that \( gXg = fXf \), hence \( g = f \) and \( f \) is primitive.

(6) clearly implies (1), completing the proof of the theorem.

Several of the above implications have been demonstrated by Numakura [6].

**Corollary 1.** Let \( e \) be a primitive idempotent of the compact mob \( X \) with zero. Then \( (Xe) \setminus N \) and \( (Xe) \cap N \) are submobs and \( (Xe) \setminus N \) is the disjoint union of the maximal (closed) groups \( (e_aXe_a) \setminus N \) where \( e_a \) runs over the nonzero idempotents of \( Xe \).
Proof. Suppose \( a, b \in (Xe) \setminus N \) and \( ab \in N \). Since \( Xe \) is a minimal non-nil left ideal, we know that \( xa = xe = xb \). Then as in the proof of (1) implies (2) we conclude \( xe = 0 \), a contradiction.

Suppose \( a, b \in (Xe) \cap N \) and \( ab \in N \). Then \( (ab)^2 \in Xab \) and \( (ab)^2 \in N \), otherwise \( ab \in N \) [5, Lemma 3]. Hence \( Xab = Xe \) by the theorem; since \( a \in Xe, Xa \subseteq Xe \). We have a right translate of \( Xa \) filling all of \( Xe \), so according to [3, Corollary 2.2.1] there is an idempotent \( f \) in \( \Gamma(b) \) which is a right unit for \( Xe \). However \( b \in N \) implies \( f = 0 \) so that \( Xe = 0 \), a contradiction.

Finally, pick \( a \in (Xe) \setminus N \); by the theorem we have \( xa = xe \) and by Lemma 3 we have \( \Gamma(a) \cap N = \emptyset \). Choose an idempotent \( f \) in \( \Gamma(a) \); then \( xe = xf \) so that \( f \) is a right unit for \( Xe \). Hence \( \Gamma(a) \) is a group, showing that \( xe \setminus N \) is the union of groups. For any nonzero idempotent \( e_a \in Xe, Xe_a = xe \) so that \( e_a \) is primitive and \( (e_a xe_a) \setminus N \) is a group. Now the maximal group \([9]\) containing \( e_a \) is contained in \( e_a xe_a \); moreover, since any group which meets \( N \) must be zero, we conclude that \( (e_a Xe_a) \setminus N \) is a maximal group. This is closed by the compactness of \( X \), completing the proof.

In [6] Numakura shows that if \( M \) is a minimal non-nil ideal, and if \( J \) is the largest ideal of \( X \) contained in \( N \), then \( M - (J \cap M) \), the difference semigroup in the sense of Rees [7], is completely simple (i.e. simple with each idempotent primitive). It follows that \( M \setminus N \) is the disjoint union of isomorphic groups, and \( M \setminus J = \bigcup [(xe_a) \setminus J] \) where \( e_a \) runs over the nonzero idempotents in \( M \). It would be of interest to know more of the multiplication in \( M \setminus J \). Corollary 1 aims in this direction. If \( E \) represents the set of primitive idempotents of the compact mob \( X \) with zero, then \( (xe Xe) \setminus J = (xe E) \setminus J \) and \( (xe E Xe) \setminus N \) is the disjoint union of groups. (In this connection, see also [1].) At this writing it is not known whether or not \( E \) must be a closed set.

As shown in [6], if \( N \) is open then there exists a nonzero primitive idempotent. According to Corollary 1, the condition that \( N \) be open may be weakened as follows:

Corollary 2. Let \( X \) be a compact mob with zero; then \( X \) contains a nonzero primitive idempotent if and only if there is a nonzero idempotent \( e \) with \( (xe Xe) \setminus N \) closed.

Proof. If \( f \) is a nonzero primitive idempotent of \( X \), then \( (xf) \setminus N \) is a maximal group and hence is closed. On the other hand, if \( (xe Xe) \setminus N \) is closed and \( e \neq 0 \), then since the set of nilpotent elements of \( xe \) is \( (xe Xe) \cap N \), we conclude from [6] that \( xe \) contains a nonzero primitive idempotent. Hence so does \( x \), completing the proof.
A five element example due to R. P. Rich [8] serves to illustrate these results; J. G. Wendel has given the following matrix representation of Rich's example:

\[
0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad l = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
r = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.
\]

This can be modified to furnish a compact connected example, as follows. Let

\[
X = \left\{ \begin{pmatrix} 0 & 0 \\ \ell & \omega \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \right\},
\]

where the entries are real numbers between \(-1\) and 1 inclusive. Here \(N\) is the totality of those matrices with main diagonal entries in the open interval \((-1, 1)\); \(J\), the largest ideal of \(X\) contained in \(N\), is the totality of those matrices with every entry lying in the open interval \((-1, 1)\). It can be shown that \(X - J\) is completely simple. If \(e\) is one of the four idempotents

\[
\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix},
\]

then \((xe)\backslash J\) consists of two disjoint two-element groups. If \(e\) is any other nonzero idempotent, then \((xe)\backslash J\) consists of one two-element group and one two-element subset whose square lies in \(J\).

**Theorem 2.** Let \(e\) be a primitive idempotent of the compact connected mob \(X\) with zero; then \((eXe)\cap N\) is dense in \(eXe\) (hence \((Xe)\cap N\) is dense in \(Xe\) and \((Xe)\cap N\) is dense in \(Xe\)).

**Proof.** We denote the compact mob \(eXe\) by \(Y\), let \(N_1 = Y \cap N\), and note that \(N_1\) is open in \(Y\) in view of Corollary 1 of Theorem 1. Denote by \(L\) the largest left ideal of \(Y\) contained in \(N_1\); since \(N_1\) is open in \(Y\), so is \(L\) [4]. Since \(L^*\) (stars denote closure) is a left ideal of \(Y\) it follows from the connectedness of \(Y\) that \(L^* \cap (Y \setminus N) \neq \emptyset\). Hence there is a nonzero idempotent in \(L^* \cap (Y \setminus N)\), and this must be \(e\). Therefore \((eXe)e = eXe \subseteq L^* \subseteq N_1^*\) so that \((eXe)\cap N\) is dense in \(eXe\).

The remainder of the theorem follows from Corollary 1 and the remarks which follow it.

In conclusion we remark that the results of this note can be extended as follows. Let \(M\) be an ideal of the mob \(X\). We define an idem-
primitive $e$ to be $M$-primitive if the only idempotents in $eXe$ either coincide with $e$ or else belong to $M$. Then by replacing $N$ by $N_M = \{a: \Gamma(a) \cap M \neq \emptyset\}$, the results obtained here for primitive idempotents hold for $M$-primitive idempotents with obvious modifications in statements and proofs; here we need not assume the existence of a zero.

**References**