A NOTE ON CONTINUOUS COLLECTIONS OF CONTINUOUS CURVES FILLING UP A CONTINUOUS CURVE IN THE PLANE

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The purpose of this note is to prove the following

THEOREM. *If G is a continuous collection of nondegenerate continuous curves filling up a compact continuous curve in the plane, then with respect to its elements as points G is either an arc or a simple closed curve.*

This theorem provides a complement to Theorem VI of [1] which states that if G is a continuous collection of nondegenerate compact continuous curves in the plane which is a compact continuum with respect to its elements as points, then G is a hereditary continuous curve such that the closure of its set of emanation points is totally disconnected.

Some notation similar to that of [1] will be used. A simple chain is a finite collection $x_1, x_2, \cdots, x_n$ of open discs (interiors of simple closed curves) such that $\bar{x}_i \cdot \bar{x}_j$ exists if and only if $|i-j| \leqq 1$ and is a closed disc (a simple closed curve plus its interior) if it does exist. A (simple) chain C is said to simply cover a set M if C^* contains M but for no proper subchain C' of C does the closure of C'^* contain M.[1] Two chains, C and C', are said to be mutually exclusive if C^* and C'^* are mutually exclusive. The chain C is said to be a closed refinement of the chain C' provided that each link of C' contains the closure of some link of C and the closure of each link of C is contained in some link of C'.

PROOF OF THEOREM. We note first that only a countable number of the elements of G contain triods. (See [5].) Denote the elements of some subset of G containing these elements by $g_1, g_2, \cdots$. Each of the remaining elements of G is an arc or a simple closed curve. If G is not an arc or a simple closed curve with respect to its elements as points, it has an emanation element. Let H denote the closure of the set of emanation elements of G. It has been noted that, by Theorem VI of [1], H is totally disconnected.

Let T be an arc of elements of G and suppose that the non-endelement t of T is an isolated element of $T \cdot H$. There is a subarc T_1 of

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[1] If G is a collection of point sets, G^* denotes the sum of its elements.

elements of T such that $T_1 \cdot H$ consists of t, a non-endelement of T_1, and possibly one or both of the endelements t_1 and t_2 of T_1. Let $\tilde{T}_1$ be $T_1 - t_1 - t_2$. Since G^* is locally connected, so is $\tilde{T}_1^* - t$. If there is a point of $t + t_1 + t_2$ at which T_1^* is not locally connected then, since G^* is locally connected, there exists an arc s in G^* which intersects $\tilde{T}_1^* - t$ and $t + t_1 + t_2$ with $s \cdot (t + t_1 + t_2)$ an endpoint of s and with s not a subset of T_1^*. This contradicts the fact that $\tilde{T}_1^* - t$ and $G^* - T_1^*$ must be mutually separated. Therefore T_1^* is a continuous curve and it follows easily that T_1^* has no cut point. Hence from Theorem 54, Chapter IV of [2] it follows that no boundary of a complementary domain of T_1^* has a cut point and therefore that every such boundary curve is a simple closed curve. Of course no two such boundary curves have more than one point in common. Since t is a non-endelement of T_1 it contains an interior point of T_1^* (Theorem 6 and Lemma 1 of [4]) and hence it is not a limit element of $G - T$. Thus $T \cdot H$ contains no isolated element and is a Cantor set if it exists.

Since G is a continuous curve, the set of emanation elements of G lying in T is dense in $T \cdot H$. Since $T \cdot H$ is closed it is not true that both the collection of arcs and the collection of simple closed curves of $T \cdot H$ are dense in $T \cdot H$. (This follows from Theorem I of [1].) Therefore there is a subarc T_2 of T intersecting H such that the endelements of T_2 are atriodic non-endelements of T which lie in $T - T \cdot H$ and either $T_2 \cdot H$ contains no element which is an arc or $T_2 \cdot H$ contains no element which is a simple closed curve.

Let V be a maximal free arc[2] of T_2 with endelements a and b in H. From the argument above (but with t not present) V^* must be locally connected. Therefore, as noted above, every complementary domain of V^* is bounded by a simple closed curve and no two such boundary curves intersect in more than one point. Since V is a free arc, no more than two of the complementary domains of V^* intersect $G^* - V^*$. Since G is continuous and a and b are limit elements of $T - V$, each of a and b lies in the boundary of a complementary domain of V^* and is an arc or a simple closed curve. Since V^* is locally connected it follows from the proof of Lemma 1 of [4] or an easy extension of our earlier arguments that each simple closed curve which is an element of V separates T_2^* in the plane and that if a or b—say a—is a simple closed curve and the sequential limiting set of a sequence of elements of $G - V$, at most a finite number of the elements of this sequence lie in the same complementary domain of a as $V^* - a$.

[2] A free arc in G is an arc of elements of G none of whose non-endelements lies in H. By a maximal free arc we shall mean a free arc whose endelements lie in H. If T is a subset of G which is an arc, we shall speak of a free arc of G which is a subset of T as a free arc of T.

Suppose $T\cdot H$ contains no arc as an element. Then it follows from the above that all the endelements of maximal free arcs in T_2 are simple closed curves. It is easy to see from the considerations above that no three such simple closed curves have the property that no one separates the other two in the plane and hence by Lemmas A and B of [1] it follows that every element of $T_2\cdot H$ is a simple closed curve. But by Lemma C of [1] no element of $T_2\cdot H$ can be an emanation element of G. Hence we may assume that $T_2\cdot H$ has no element which is a simple closed curve.

It follows in a similar manner that only a finite number of the maximal free arcs of T_2 can contain simple closed curves or, in fact, continuous curves which separate the plane as elements and hence we may consider without essential loss of generality that T_2 contains no element which separates the plane and which is in a free arc of T_2.

Since each element x of a free arc of T_2 is an endelement of some free arc U of T_2 and thus a subset of a boundary simple closed curve of a complementary domain of U^*, x is an arc and it follows from Theorem 3 of [3] that the sum of the elements of every free arc of T_2 is a closed disc.

Let the element x_1 of H be an endelement of a maximal free arc V_1 of T_2. There are a simple chain C_1 of link diameter less than 1 and two open discs D_1 and D_2 such that (1) C_1^* does not intersect g_1 (the first triodic element of G as defined above) and C_1 simply covers x_1 but does not cover V_1^*, (2) C_1 is made up of the links of two simple chains, C_{11} and C_{12}, each of which has at least five links and which have only one link in common, (3) if d is an endlink of C_1, $\bar{d}-d$ and x_1 have only one point in common and if d is a non-endlink, $\bar{d}-d$ and x_1 have only two points in common, (4) D_1 and D_2 contain the endpoints of x_1, (5) for each i $(i=1, 2)$, $\overline{D}_i-D_i$ has only one point in common with x_1 and (6) the $\overline{D}_i$ $(i=1, 2)$ lie in distinct endlinks of C_1 with neither intersecting the closure of any other link of C_1.

Since G^* is locally connected at every point of x_1 there is a subarc K_1 of $T_2-V_1+x_1$ containing x_1 such that every element of K_1 is covered by C_1 and intersects both D_1 and D_2 and if A_1, A_2, and A_3 are any consecutive links of C_1 in that order there is an arc $\tilde{t}$ lying in $G^*\cdot[A_2-A_2\cdot\mathrm{Cl}\ (A_1+A_3)]$ whose endpoints are separated in $A_2-A_2\cdot(A_1+A_3)$ by the common part of $A_2-A_2\cdot(A_1+A_3)$ and any arc in K_1^* which intersects both $\overline{A}_1$ and $\overline{A}_3$. Since there is a domain U of elements of G with U containing x_1, with the closure of U^* lying in C_1^*, and with every element of U containing an arc irreducible between $\overline{D}_1$ and $\overline{D}_2$, and since every such domain contains elements of $G-T_2$, there is a sequence of components of $K_1^*\cdot[C_1^*-(D_1+D_2)]$ converging to $x_1 - x_1 \cdot (D_1 + D_2)$ each element of which intersects both $\overline{D}_1$ and $\overline{D}_2$.

There are open discs D_1' and D_2' containing $\overline{D}_1$ and $\overline{D}_2$ respectively such that for each i $(i=1, 2)$, $\overline{D}_i' - D_i'$ and x_1 have only one point in common and the $\overline{D}_i'$ $(i=1, 2)$ lie in distinct endlinks of C_1 with each intersecting the closure of no other link of C_1. Since K_1 is continuous and an arc with respect to its elements and each of its elements intersects both D_1 and D_2, there is an element z of K_1 containing two mutually exclusive arcs, z_1 and z_2, each irreducible between $\overline{D}_1'$ and $\overline{D}_2'$ and lying in different components of $K_1^* \cdot [C_1^* - (D_1 + D_2)]$. If z belongs to H there is an element x of $H \cdot K_1$ which is an endelement of a maximal free arc of K_1 and contains two mutually exclusive arcs, one intersecting both endlinks of C_1, the other covered by C_{11} or C_{12} and intersecting both of its endlinks.[3] If z is not in H it is a nonendelement of a maximal free arc Q of K_1. If Q_i $(i=1, 2)$ denotes the component of $Q^* \cdot [C_1^* - (D_1' + D_2')]$ containing z_i, Q_i contains two arcs, q_{i1} and q_{i2}, each lying on the boundary simple closed curve of Q^* and irreducible between $\overline{D}_1'$ and $\overline{D}_2'$. Since if A_1, A_2, and A_3 are any consecutive links of C_1 in that order, there is an arc $\tilde{t}$ lying in $G^* \cdot [A_2 - A_2 \cdot \mathrm{Cl}\ (A_1 + A_3)]$ whose endpoints are separated in $A_2 - A_2 \cdot (A_1 + A_3)$ by every arc of Q^* intersecting both $\overline{A}_1$ and $\overline{A}_3$, none of the q_{ij} $(i, j=1, 2)$ contains an arc t' which intersects both $\overline{A}_1$ and $\overline{A}_3$ but does not intersect either endelement of Q. This follows from the fact that such a t' would lie on the boundary simple closed curve of Q^* and contain no limit point of $G^* - Q^*$, which is inconsistent with the existence of the arc $\tilde{t}$. Therefore, since each endelement of Q intersects both $\overline{D}_1'$ and $\overline{D}_2'$, one such endelement contains two mutually exclusive arcs, one covered by C_1 and intersecting each of its links, the other covered by either C_{11} or C_{12} and intersecting each of its links.

In any case there is an element x_2 $(\neq x_1)$ of $H \cdot K_1$ which is an endelement of a maximal free arc V_2 of K_1 and contains two mutually exclusive arcs, z and z', z covered by C_1 and intersecting each of its links and z' covered by a chain C_1' $(=C_{11}$ or $C_{12})$ and intersecting each of its links. There are mutually exclusive simple chains C_2 and C_2' of link diameter less than $1/2$ simply covering z and z' respectively, C_2 a closed refinement of C_1, C_2' a closed refinement of C_1', and there is a subarc K_2 of $K_1 - V_2 + x_2$ containing x_2 such that every element of it contains two arcs, one covered by C_2 and intersecting each of its links, the other covered by C_2' and intersecting each of its links. Thus

[3] Clearly we could, if we chose, require that each of these arcs intersects both endlinks of C_1.

by repetition of the above argument, noting that a proper subarc of x_1 could have been used in lieu of x_1 and neglecting the chain C_2' in establishing the chains C_3 and C_3', it follows readily that there exist a sequence C_2', C_3', $\cdots$ of mutually exclusive chains and a sequence K_2, K_3, $\cdots$ of arcs of elements of T such that (1) $K_2 \cdot K_3 \cdot \cdots$ is an element g of G, (2) for each $i > 1$, C_i' is a closed refinement of C_{11} or C_{12} of link diameter less than $1/i$, and (3) K_{i+1} is a subarc of K_i not containing g_{i+1} such that if $j \leqq i$, each element of K_{i+1} contains an arc covered by C_j' and intersecting each of its links. The set g thus contains infinitely many mutually exclusive continua all with diameter greater than some number. But g does not contain a triod and thus is an arc or a simple closed curve. Hence the assumption that G has an emanation point leads to a contradiction and the theorem is proved.

References

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