A CORRECTION TO "ON A CONJECTURE CONCERNING DOUBLY STOCHASTIC MATRICES"

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Professor A. Horn has kindly pointed out to me in a letter dated February 15, 1953, that the proof contained in *On a conjecture concerning doubly stochastic matrices*, Proc. Amer. Math. Soc. vol. 3, pp. 511–513, is incorrect. The extension described on lines 3 et seq. on p. 513 cannot always be carried out. For whatever interest that remains it might be remarked that the theorem is true for 3×3 matrices. Professor Horn has given a counterexample for the case of 4×4 matrices. This counterexample settles in the negative the conjecture raised at the end of the paper. Drs. A. J. Hoffman and Truman Botts had previously settled the conjecture cited. Professor Horn's counterexample and proof follow: Let

$$P^1 = \begin{pmatrix} 1/4 & 0 & 3/8 & 3/8 \\ 1/4 & 0 & 3/8 & 3/8 \\ 1/4 & 1/2 & 5/32 & 3/32 \\ 1/4 & 1/2 & 3/32 & 5/32 \end{pmatrix}, \qquad P^3 = \begin{pmatrix} 1/4 & 0 & 1/4 & 1/2 \\ 1/4 & 0 & 1/2 & 1/4 \\ 1/4 & 3/4 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}.$$

There is no doubly stochastic (d.s.) matrix P^2 with $P^1 = P^2P^3$. Because if there were,

$$\begin{aligned} &\overset{1}{P}_{12} = 0 \rightarrow \overset{2}{P}_{13} = \overset{2}{P}_{14} = 0 \quad \text{and} \quad \overset{1}{P}_{22} = 0 \rightarrow \overset{2}{P}_{23} = \overset{2}{P}_{24} = 0, \\ &\overset{1}{P}_{13} = 3/8 \quad \text{and} \quad \overset{1}{P}_{14} = 3/8 \rightarrow \overset{2}{P}_{11} = \overset{2}{P}_{12} = 1/2, \\ &\overset{1}{P}_{23} = 3/8 \quad \text{and} \quad \overset{1}{P}_{24} = 3/8 \rightarrow \overset{2}{P}_{21} = \overset{2}{P}_{22} = 1/2. \end{aligned}$$

Therefore $P^2_{31} = P^2_{41} = P^2_{32} = P^2_{42} = 0$.

But then there is the contradiction: $P^1_{33} = 5/32 = (P^2_{34})/4$, $P^1_{34} = 3/32 = (P^2_{34})/4$. In order to show $P^1x < P^3x$ for all x, choose the following basis:

$$\begin{aligned} x^1 &= (0, 0, 4, -4), & x^3 &= (1, 1, 1, 1), \\ x^2 &= (0, 4/3, 0, 0), & x^4 &= (-3, 1, 1, 1). \end{aligned}$$

If $x = \alpha x^1 + \beta x^2 + \gamma x^3 + \delta x^4$, then:

$$\begin{aligned} P^1x &= \alpha P^1x^1 + \beta P^1x^2 + \gamma P^1x^3 + \delta P^1x^4 \\ &= \alpha\cdot(0, 0, 1/4, -1/4) + \beta\cdot(0, 0, 2/3, 2/3) + \gamma\cdot(1, 1, 1, 1), \end{aligned}$$

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$$P^3x = \alpha P^3x^1 + \beta P^3x^2 + \gamma P^3x^3 + \delta P^3x^4$$
$$= \alpha\cdot(-1, 1, 0, 0) + \beta\cdot(0, 0, 1, 1/3) + \gamma\cdot(1, 1, 1, 1).$$

Since $x < y \leftrightarrow x + \gamma(1, 1, 1, 1) < y + \gamma(1, 1, 1, 1)$ it remains only to prove

$$(0, 0, \alpha/4 + (2/3)\beta, -\alpha/4 + (2/3)\beta) < (-\alpha, \alpha, \beta, \beta/3) \quad \text{for all } \alpha, \beta.$$

Now if $a_1 \geqq a_2 \geqq a_3 \geqq a_4$ and if $\sum_1^4 a_i = \sum_1^4 b_i$, then $(b_1, b_2, b_3, b_4) < (a_1, a_2, a_3, a_4)$ if and only if $a_4 \leqq b_i \leqq a_1$, $1 \leqq i \leqq 4$, and

$$a_3 + a_4 \leqq \begin{Bmatrix} b_1 + b_2 \\ b_1 + b_3 \\ b_2 + b_3 \end{Bmatrix} \leqq a_1 + a_2.$$

So it remains to be shown:

$$\text{(1)} \quad \min(-\alpha, \alpha, \beta, \beta/3) \leqq \begin{Bmatrix} 0 \\ \alpha/4 + (2/3)\beta \\ -\alpha/4 + (2/3)\beta \end{Bmatrix} \leqq \max(-\alpha, \alpha, \beta, \beta/3);$$

(2) The sum of the 2 smallest of $-\alpha, \alpha, \beta, \beta/3 \leqq \begin{Bmatrix} 0 \\ \alpha/4 + (2/3)\beta \end{Bmatrix}$

$\leqq$ the sum of the 2 largest of $-\alpha, \alpha, \beta, \beta/3$.

For (1):

$$\max(-\alpha, \alpha, \beta, \beta/3) \geqq \begin{Bmatrix} \alpha \\ -\alpha \end{Bmatrix} \geqq \min(-\alpha, \alpha, \beta, \beta/3),$$
$$\max(-\alpha, \alpha, \beta, \beta/3) \geqq 0 \geqq \min(-\alpha, \alpha, \beta, \beta/3).$$

Also

$$\alpha/4 + (2/3)\beta = (1/4)(\alpha) + (5/8)(\beta) + (1/8)(\beta/3),$$

a mean of $\alpha, \beta, \beta/3$,

$$-\alpha/4 + (2/3)\beta = (1/4)(-\alpha) + (5/8)(\beta) + (1/8)(\beta/3),$$

a mean of $-\alpha, \beta, \beta/3$.

For (2): Sum of the two largest of $-\alpha, \alpha, \beta, \beta/3 \geqq \alpha - \alpha = 0 \geqq$ sum of two smallest of $-\alpha, \alpha, \beta, \beta/3$. Also $\alpha/4 + (2/3)\beta = (5/16)(0) + (1/4)(\alpha + \beta/3) + (7/16)((4/3)\beta)$, a mean of pair-sums from $-\alpha, \alpha, \beta, \beta/3$.

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