UNIQUENESS THEOREMS FOR A CLASS OF GENERALIZED TRIGONOMETRICAL SERIES[1]

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1. One of the applications which Cauchy made of the theory of residues was to the expansion of a function, $f(x)$, defined say in $(-\pi, \pi)$, as a series of the form

$$a_0 + \sum_1^{\infty} (a_n \cos \mu_n x + b_n \sin \mu_n x) \tag{1}$$

where

$$0 = \mu_0 < \mu_1 < \mu_2 < \cdots$$

are the non-negative roots of the equation in z

$$z + h \tan \pi z = 0 \qquad (h > 0).$$

The formulae for the coefficients in (1) can be written

$$a_n + ib_n = K_n \int_{-\pi}^{\pi} f(t) e^{i\mu_n t} dt \qquad (n = 1, 2, \cdots) \tag{2}$$

where

$$K_n = \frac{2\mu_n}{2\pi\mu_n - \sin 2\pi\mu_n},$$

and

$$a_0 = K_0 \int_{-\pi}^{\pi} f(t) dt \tag{3}$$

where

$$K_0 = \frac{h}{2(1 + \pi h)} \cdot$$

The modern treatment of the series (1) is due to Fejes [1]. We shall call (1) the F.A. series of $f(x)$ if equations (2) and (3) are satisfied.

In this note we are concerned with uniqueness theorems for convergent or summable series of the type (1). An important difference

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[1] This note is a simplified account of part of a thesis which was awarded the Ph.D. degree at Queen's University, Belfast, in 1953.

between the results obtained and those known for trigonometric series arises from the fact that there is a linear equation connecting the coefficients when (1) is an F.A. series. This equation (see Fejes [1]) is

$$a_0 = -\sum_1^\infty a_n \cos \mu_n \pi. \tag{4}$$

It is therefore desirable to enunciate our theorems in terms of a series without a constant term. We shall call

$$\sum_1^\infty (a_n \cos \mu_n x + b_n \sin \mu_n x) \tag{5}$$

the modified F.A. series (M.F.A. series) of $g(x)$ if there is a constant a_0 (which is then necessarily given by (4)) such that (1) is the F.A. series of $g(x)+a_0$. We shall then write

$$g(x) + a_0 \sim a_0 + \sum_1^\infty (a_n \cos \mu_n x + b_n \sin \mu_n x).$$

We use the notation

$$A_n(x) = a_n \cos \mu_n x + b_n \sin \mu_n x, \qquad P(r, x) = \sum A_n(x) r^n,$$

and denote the upper or lower limits of $P(r, x)$, as $r \to 1-$, by $P^*(x)$, $P_*(x)$ respectively. We shall prove

THEOREM 1. *If* (5) *converges in* $(-\pi, \pi)$, *except at an enumerable set, to a function* $f(x) \in L(-\pi, \pi)$, *then it is the M.F.A. series of* $f(x)$.

THEOREM 2. *If* (i) $a_n = o(n)$, $b_n = o(n)$, *and*
(ii) $P^*(x)$, $P_*(x)$ *are finite in* $-\pi \leqq x \leqq \pi$ *and integrable, then* (5) *is the sum of an M.F.A. series and of the series*

$$k \sum_1^\infty K_n \sin \mu_n \pi \sin \mu_n x,$$

where k is some constant.

THEOREM 3. *If, in Theorem* 2, $P^*(x) = P_*(x) = 0$ *almost everywhere, then* $a_n = b_n = 0$ *for all n.*

A more elaborate argument[2] than is given in this note will prove that in Theorem 2 we may have an exceptional enumerable set at the points of which $P_*(x)$, $P^*(x)$ are not required to be finite, but at which

[2] This argument is given in my Ph.D. thesis.

$$\lim_{r\to 1}(1-r)P(r,x)=0.$$

2. We shall denote by $S_n(f;x)=S_n(x)$ the nth partial sum of the F.A. series of $f(x)$. Corresponding to any integrable function $f(x)$ we define an associated function $\bar{f}(x)$, of period 4π, by the equations

$$\bar{f}(x)=\begin{cases} f(x) & (-\pi\leqq x\leqq\pi),\\ -f(x-2\pi) & (\pi<x<3\pi).\end{cases}$$

In this notation, the results of S. Verblunsky [2] concerning $S_n(f;x)$ and its derivatives may be written

$$S_n(f;x)=S^*_{2n}(\bar{f};x)+\epsilon_n^{(1)}(x), \tag{6}$$

$$S'_n(f;x)=S^{*\prime}_{2n}(\bar{f};x)-\frac{h}{2\pi^2}\int_{-\pi}^{\pi}f(t)\,\frac{\xi\sin n\xi}{\sin\xi/2}\,dt+\epsilon_n^{(2)}(x), \tag{7}$$

$$S''_n(f;x)=S^{*\prime\prime}_{2n}(\bar{f};x)-\frac{h}{2\pi^2}\int_{-\pi}^{\pi}f(t)\,\frac{d}{d\xi}\left(\frac{\xi\sin n\xi}{\sin\xi/2}\right)dt+\epsilon_n^{(3)}(x), \tag{8}$$

where $\epsilon_n^{(1)}(x)\to 0$ uniformly for $|x|\leqq\pi$, $\epsilon_n^{(2)}(x)$ and $\epsilon_n^{(3)}(x)\to 0$ for $|x|\leqq\pi$, $\xi=x-t$, and $S^*_n(f;x)=S^*_n(x)$ denotes the nth partial sum of the Fourier series of $f(x)$.

LEMMA 1. *We have*

$$\lim_{r\to 1}(1-r)\sum S^*_{2n}(\bar{f};x)r^n=\lim_{r\to 1}(1-r)\sum S^*_n(\bar{f};x)r^n$$

when either side exists. Moreover, we may replace $S^*_{2n}(x)$, $S^*_n(x)$ *by* $S^{*\prime}_{2n}(x)$, $S^{*\prime}_n(x)$, *or by* $S^{*\prime\prime}_{2n}(x)$, $S^{*\prime\prime}_n(x)$, *and in each of the results so obtained we may replace* lim *by* lim sup *or by* lim inf *throughout.*

For

$$\begin{aligned}\lim_{r\to 1}(1-r)\sum S^*_{2n}(x)r^n &= \lim_{r\to 1}(1-r^2)\sum S^*_{2n}(x)r^{2n}\\ &= 2\lim_{r\to 1}(1-r)\sum S^*_{2n}(x)r^{2n}\\ &= \lim_{r\to 1}(1-r)\sum S^*_{2n}(x)r^{2n}\\ &\quad+\lim_{r\to 1}r(1-r)\sum S^*_{2n-1}(x)r^{2n-1}\end{aligned}$$

since, for a function of the type $\bar{f}(x)$, we have $S^*_{2n}(x)=S^*_{2n-1}(x)$. It follows that

$$\lim_{r\to 1}(1-r)\sum S^*_{2n}(x)r^n = \lim_{r\to 1}(1-r)\sum S^*_n(x)r^n$$

when either side exists. The other cases may be dealt with similarly.

3. We shall denote by $\mathcal{J}'\{f(x)\}$ the series obtained by differentiating the F.A. series of $f(x)$, and by $\mathcal{J}''\{f(x)\}$ the series obtained by differentiating $\mathcal{J}'\{f(x)\}$. The Poisson-sums of $\mathcal{J}''\{f(x)\}$ shall be denoted by $\mathcal{P}(r, x)$, $\mathcal{P}^*(x)$, $\mathcal{P}_*(x)$. We shall also use the notation

$$\Delta^2 f(x, t) = f(x+t) + f(x-t) - 2f(x),$$

$$D^{*2}f(x) = \limsup_{t\to 0}\frac{\Delta^2 f(x, t)}{t^2},$$

$$D^2_* f(x) = \liminf_{t\to 0}\frac{\Delta^2 f(x, t)}{t^2}.$$

We may now prove

LEMMA 2. *If at a point x_0 ($|x_0| < \pi$)*

$$\lim_{t\to 0}\frac{f(x_0+t) - f(x_0-t)}{2t} = l$$

where l is not necessarily finite, then $\mathcal{J}'\{f(x)\}$ is summable-P at x_0 to l.

We consider the Poisson-limit of $S_n'(f; x)$. Since the second term on the right-hand side of (7) is $o(1)$ for $|x| < \pi$ the result follows from Lemma 1 and from the theory of Fourier series [3, p. 52] applied to the function $\bar{f}(x)$.

LEMMA 3. *If $f(-\pi+0)$ and $f(\pi-0)$ exist, and if*

$$\lim_{t\to 0}\frac{f(-\pi+t) + f(\pi-t)}{2t} = l$$

where l is not necessarily finite, then $\mathcal{J}'\{f(x)\}$ is summable-P at $x=\pi$ to $hf(-\pi+0)-l$, and at $x=-\pi$ to $l-hf(\pi-0)$.

We must reconsider the second term on the right-hand side of (7). At $x=\pi$ it equals $-(h/\pi)S^*_{2n}(\bar{g}; \pi)$, where $g(x)=(\pi-x)f(x)$, and so has the Poisson-limit

$$-\frac{h}{2\pi}[\bar{g}(\pi-0) + \bar{g}(\pi+0)] = hf(-\pi+0).$$

Using Lemma 1 and the theory of Fourier Series [3, p. 52], we find that the first term on the right-hand side of (7) has, at $x=\pi$, the

Poisson-limit $-l$. Hence $\mathcal{J}'\{f(x)\}$ is summable-P at $x=\pi$ to $hf(-\pi+0)-l$, and similarly at $x=-\pi$ to $l-hf(\pi-0)$.

COROLLARY 1. *At* $x=\pi$, $\mathcal{J}'\{1\}$ *is summable-P to* $-\infty$, *and at* $x=-\pi$ *to* ∞.

LEMMA 4. *If* $f(x)$ *is continuous at* $x=x_0$ ($|x_0|<\pi$), *then*

$$D^{*2}f(x_0) \geqq \mathcal{P}_*(x_0), \qquad D_*^2 f(x_0) \leqq \mathcal{P}^*(x_0).$$

Consider the second term on the right-hand side of (8). It equals

$$\begin{aligned}(9)\qquad & -\frac{h}{2\pi^2}\int_{-\pi}^{\pi} f(t)\xi\frac{d}{d\xi}\left(\frac{\sin n\xi}{\sin \xi/2}\right)dt - \frac{h}{2\pi^2}\int_{-\pi}^{\pi} f(t)\,\frac{\sin n\xi}{\sin \xi/2}\,dt \\ & = -\frac{h}{\pi}S_{2n}^{*\prime}(\bar{g};\,x) - \frac{h}{\pi}S_{2n}^{*}(\bar{f};\,x)\end{aligned}$$

where $g(t)=(x-t)f(t)$. The second term in (9) has the Poisson-limit $-(h/\pi)f(x_0)$ at $x=x_0$. To see that the first term has equal but opposite Poisson-limit we note that

$$\lim_{t\to 0}\frac{\bar{g}(x_0+t)-\bar{g}(x_0-t)}{2t} = \lim_{t\to 0}\frac{-tf(x_0+t)-tf(x_0-t)}{2t} = -f(x_0).$$

The lemma now follows from the theory of Fourier series [**4**, p. 268] applied, in conjunction with Lemma 1, to the first term on the right-hand side of (8).

4. LEMMA 5. *If* (5) *converges uniformly in* $(-\pi,\pi)$ *to* $f(x)\in L(-\pi,\pi)$, *then it is the M.F.A. series of* $f(x)$.

By straightforward integration, remembering that $\mu_n+h\tan\pi\mu_n=0$ for all n, we find that, if n is nonzero,

$$\int_{-\pi}^{\pi}\sin\mu_m t\sin\mu_n t\,dt = \begin{cases} 0 & (m\neq n), \\ \dfrac{1}{K_n} & (m=n). \end{cases}$$

Hence

$$\begin{aligned}\int_{-\pi}^{\pi}\cos\mu_m t\cos\mu_n t\,dt &= \frac{2\sin\mu_n\pi\cos\mu_m\pi}{\mu_n} + \frac{\mu_m}{\mu_n}\int_{-\pi}^{\pi}\sin\mu_m t\sin\mu_n t\,dt \\ &= \frac{2\sin\mu_n\pi\cos\mu_m\pi}{\mu_n} + \theta_{m,n}\end{aligned}$$

where

$$\theta_{m,n} = \begin{cases} 0 & (m \neq n), \\ \dfrac{1}{K_n} & (m = n). \end{cases}$$

It then follows, by the term by term integration of a uniformly convergent series, that for any constant c

$$K_n \int_{-\pi}^{\pi} [f(t) + c] \cos \mu_n t dt = \frac{2cK_n \sin \mu_n \pi}{\mu_n} + \frac{2K_n \sin \mu_n \pi}{\mu_n} \sum_{\nu=1}^{\infty} a_\nu \cos \mu_\nu \pi + a_n.$$

Choosing $c = -\sum_1^\infty a_n \cos \mu_n \pi$, the right-hand side equals a_n. Further

$$K_n \int_{-\pi}^{\pi} [f(t) + c] \sin \mu_n t dt = b_n.$$

It follows that $f(x)+c \sim c + \sum_1^\infty (a_n \cos \mu_n x + b_n \sin \mu_n x)$, and the lemma is proved.

PROOF OF THEOREM 1. The argument of the Cantor-Lebesgue Theorem [3, p. 267] applies to show that, since $A_n(x) \to 0$ p.p. in $(-\pi, \pi)$, $a_n = o(1)$ and $b_n = o(1)$. Hence

$$F^*(x) = -\sum_1^\infty A_n(x)/\mu_n^2$$

converges uniformly in $(-\pi, \pi)$. If $F(x) = F^*(x) + c$, where $c = \sum_1^\infty a_n \cos \mu_n \pi / \mu_n^2$, we have, by Lemma 5,

$$K_n \int_{-\pi}^{\pi} F(t) e^{i\mu_n t} dt = -\frac{a_n + ib_n}{\mu_n^2} \quad (n = 1, 2, \cdots). \tag{10}$$

Now $\mathcal{J}''\{F(x)\}$ is summable-P to $f(x)$ and so, by Lemma 4,

$$D^{*2}F(x) \geqq f(x) \geqq D_*^2 F(x)$$

in $(-\pi, \pi)$ except in an enumerable set. Since $a_n = o(1)$, $b_n = o(1)$, $F(x)$ is smooth. It follows [3, p. 276, Theorem (iv)] that

$$F(x) = \int_{-\pi}^{x} dy \int_{-\pi}^{y} f(t) dt + Ax + B$$

where A, B are constants. We may now integrate by parts twice on the left-hand side of equation (10). Equating real and imaginary parts, and remembering that $F(\pi)+F(-\pi)=0$ by (4), we get

$$(11)\qquad a_n = K_n \int_{-\pi}^{\pi} f(t) \cos \mu_n t dt - K_n \cos \mu_n \pi \int_{-\pi}^{\pi} f(t) dt,$$

$$(12)\qquad b_n = K_n \int_{-\pi}^{\pi} f(t) \sin \mu_n t dt - \lambda K_n \sin \mu_n \pi,$$

where $\lambda = F'(\pi)+F'(-\pi)+h[F(\pi)-F(-\pi)]$. Writing

$$(13)\qquad c' = \frac{h}{2}\int_{-\pi}^{\pi} f(t) dt,$$

the equations (11) and (12) give

$$(14)\qquad K_n \int_{-\pi}^{\pi} [f(t)+c'] \cos \mu_n t dt = a_n,$$

$$(15)\qquad K_n \int_{-\pi}^{\pi} [f(t)+c'] \sin \mu_n t dt = b_n + \lambda K_n \sin \mu_n \pi.$$

The left-hand side of (15) is $o(1)$, and we have seen that $b_n=o(1)$. Now $K_n \to 1/\pi$ and

$$\sin \mu_n \pi = \pm \frac{\mu_n}{(h^2+\mu_n^2)^{1/2}}\cdot$$

Hence $\lambda=0$. Lastly, (13) implies that

$$(16)\qquad c' = K_0 \int_{-\pi}^{\pi} [f(t)+c'] dt.$$

The theorem now follows from (14), (15), and (16).

5. Proof of Theorem 2. We have

$$\sum_1^\infty \frac{A_n(x)}{n^2} r^n = \int_0^r \frac{d\rho}{\rho} \int_0^\rho \frac{P(t, x)}{t} dt = I(r) \quad \text{(say)}.$$

Now

$$P(t, x)/t = \sum_1^\infty A_n(x) t^{n-1}$$

which is a continuous function of t in $0 \leqq t < 1$. Since $P^*(x)$, $P_*(x)$ are finite in $(-\pi, \pi)$, $P(t, x)/t$ is bounded as $t \to 1$ and so $\lim_{r\to 1} I(r)$ exists. Hence $\lim_{r\to 1} \sum_1^\infty A_n(x) r^n/n^2$ exists for all x in $(-\pi, \pi)$. It follows by Tauber's Theorem that $\sum A_n(x)/n^2$ is convergent for all x in $(-\pi, \pi)$

and so therefore is $\sum A_n(x)/\mu_n^2$ since $\sum_1^\infty (A_n(x)/n^2 - A_n(x)/\mu_n^2)$ is uniformly convergent. Thus $F^*(x) = -\sum_1^\infty A_n(x)/\mu_n^2$ is finite in $(-\pi, \pi)$ and can be written as $\sum_1^\infty (\alpha_n \cos \mu_n x + \beta_n \sin \mu_n x)$ $(\alpha_n = o(1/n)$, $\beta_n = o(1/n))$. Now, $\mu_n = n - 1/2 + O(1/n)$, and so

$$F^*(x) = \sum_1^\infty \{\alpha_n \cos (n - 1/2)x + \beta_n \sin (n - 1/2)x\} + G(x)$$

where $G(x)$ is bounded. The series on the right-hand side is the sum, at $t = x/2$, of the Fourier series of a function $g(t) \in L^2(-\pi, \pi)$. Since it converges for t in $(-\pi/2, \pi/2)$ it must converge to $g(t)$ p.p. in $(-\pi/2, \pi/2)$. Hence $F^*(x)$ is integrable in $(-\pi, \pi)$, and by Theorem 1 the equation (10) is true with $F(x) = F^*(x) + c$. Now by the argument of Zygmund [**3**, p. 301] it can be shown that $F(x)$ is continuous. Hence [**3**, p. 276, Theorem (iv)] we have

$$F(x) = \int_{-\pi}^{x} dy \int_{-\pi}^{y} f(t)dt + Ax + B$$

where A, B are constants and $f(x) = \text{Min}\,[D^{*2}F(x), P^*(x)]$. Equations (14) and (15), with c' defined as in (13), may now be deduced from (10) exactly as in the proof of Theorem 1. Writing $k = -\lambda$ we see that Theorem 2 is now proved.

6. Proof of Theorem 3. By Lemma 4, $D^{*2}F(x) \geqq 0$ and so $f(x) = 0$ p.p. in $(-\pi, \pi)$, where $F(x)$, $f(x)$ are defined as in the proof of Theorem 2. Hence in this case Theorem 2 shows that (5) reduces to

$$k \sum_1^\infty K_n \sin \mu_n \pi \sin \mu_n x.$$

This series is $\mathcal{J}'\{-k/2\}$ which, by Corollary 1, has infinite Poisson sum at $x = -\pi$ and $x = \pi$. Since $P^*(x)$, $P_*(x)$ were supposed finite in $-\pi \leqq x \leqq \pi$ we must have $k = 0$. The result follows.

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Bibliography

1. L. Fejes, *Über die Fouriersche Reihe der Abkühlung*, Acta Univ. Szeged. vol. 11 (1946–48) pp. 28–37.

2. S. Verblunsky, *Sur une classe de séries exponentielles de Cauchy*, Bull. Sci. Math. vol. 76 (1952) pp. 85–96.

3. A. Zygmund, *Trigonometrical series*, 1935.

4. A. Rajchman and A. Zygmund, Math. Zeit. vol. 25 (1926) pp. 261–273 (268).

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