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ON THE SPECTRA OF COMMUTATORS[1]

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1. The following theorems will be proved:

(*) *Let A and B denote bounded operators in a Hilbert space and suppose that the commutator*

$$C = AB - BA \tag{1}$$

satisfies the commutation relation

$$AC = CA. \tag{2}$$

Then there exists a sequence of bounded operators $C_1, C_2, \cdots$ such that

$$\|C - C_n\| \to 0 \quad \text{and} \quad \text{sp}\,(C_n) \to 0 \text{ as } n \to \infty. \tag{3}$$

(**) *If, in addition to the assumption* (2) *of* (*), *the relation*

$$BC = CB \tag{4}$$

also holds, then the assertion (3) *can be improved to*

$$\text{sp}\,(C) \textit{ consists of } 0 \textit{ alone.} \tag{5}$$

For the terminology see, e.g., [**3**, pp. 9–10]. It is understood that sp (D) denotes the set of points in the spectrum of an operator D. Thus, the relation sp $(C_n)\to 0$ means that for any $\epsilon>0$ there is a number N_ϵ such that the spectrum of C_n is contained in the disk $|\lambda|<\epsilon$ whenever $n>N_\epsilon$.

As a consequence of (**) one obtains the

COROLLARY. *If A is a bounded operator in a Hilbert space which com-*

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mutes with AA^*-A^*A, *then necessarily* $AA^*-A^*A=0$, *so that* A *is normal.*

In order to prove the corollary one need only note that $C=AA^*-A^*A$ is Hermitian, so that $AC=CA$ implies $A^*C=CA^*$. Thus, if $B=A^*$, relations (2) and (4) hold. Consequently, by (**), the spectrum of the Hermitian operator C consists of 0 alone, and hence $C=0$.

It is known that if A is bounded and commutes with both AA^* and A^*A, then A must be normal [**1**, p. 727]; the corollary thus represents a certain refinement of this result.

Relation (2) is known to imply that $\lambda=0$ is in the spectrum of C and, in fact, that $Cf_n\to 0$, as $n\to\infty$, holds for some sequence of elements f_n satisfying $\|f_n\|=1$; Halmos [**2**, pp. 192–193]. Furthermore, if A and B denote finite matrices satisfying (1) and (2), then the assertion (3) of (*) clearly implies the statement (5). In addition, it is known that if A and B are bounded (not necessarily finite) and if A is normal, then (2) implies even $C=0$; [**4**, p. 359].

Whether condition (2) alone is sufficient to guarantee (5) whenever A and B are bounded is not known; a conjecture of Kaplansky cited by Halmos [**2**, p. 192] would imply that this is indeed the case.

2. **Proof of** (*). The conditions (1) and (2) imply $A^nB-BA^n=nA^{n-1}C$ for every positive integer n; cf. [**5**] and [**2**, p. 192]. Since C of (1) is unchanged (and (2) remains valid) if A is replaced by $A+\alpha I$, where α is any complex number, it can be supposed that A is nonsingular; cf. [**7**]. Consequently,

$$C - C_n = n^{-1}AB, \quad \text{where} \quad C_n = -A^{-n}(n^{-1}AB)A^n. \tag{6}$$

It is clear that $\|n^{-1}AB\|\to 0$ when $n\to\infty$ and that $\operatorname{sp}(C_n)=\operatorname{sp}(-n^{-1}AB)$. Hence (3) holds and the proof of (*) is complete.

3. **Proof of** (**). Put $M=n^{-1}AB$ and $N=C_n=-A^{-n}(n^{-1}AB)A^n$, so that (6) becomes $C=M+N$. Then the assumptions (2) and (4) imply $MN=NM$ and hence

$$C^m = \sum_{k=0}^{m}\binom{m}{k} M^{m-k}N^k, \tag{7}$$

for an arbitrary positive integer m. Since $M^{m-k}=n^{k-m}(AB)^{m-k}$ and $N^k=(-n)^{-k}A^{-n}(AB)^kA^n$, it is clear from (7) that

$$\|C^m\| \leqq \sum_{k=0}^{m}\binom{m}{k}\|M^{m-k}\|\,\|N^k\| \leqq a_n(2c/n)^m, \tag{8}$$

where $a_n=\|A^{-n}\|\|A^n\|$ and $c=\|AB\|$. Choose $n>2c$ and keep n fixed.

It then follows from (8) that

$$\|C^m\| \to 0, \qquad m \to \infty. \tag{9}$$

By taking conjugates in (1), (2), and (4), one can show by a similar argument that

$$\|C^{*m}\| \to 0, \qquad m \to \infty. \tag{10}$$

In order to prove (5), suppose, if possible, the contrary; that is, suppose $\lambda \neq 0$ is in the spectrum of C. Since each of the relations (2) and (4) remain valid if, in (1), A is replaced by αA for any complex number α, it can clearly be supposed that $\lambda = 1$ occurs in the spectrum of C. In view of the Toeplitz criterion (cf., e.g., [6, p. 138]), either $(C-I)(C-I)^*$ or $(C-I)^*(C-I)$ is *not* positive definite. Thus, there exists a sequence of elements f_n such that $\|f_n\| = 1$ and either $Cf_n - f_n \to 0$ or $C^*f_n - f_n \to 0$ as $n \to \infty$. In either case, in view of (9) and (10), one obtains a contradiction and the proof of (**) is now complete.

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