ALGEBRAS OF DIFFERENTIABLE FUNCTIONS[1]

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1. Let M be a compact differentiable manifold of class C^r, $1 \leqq r < \infty$, and let $C^r(M)$ be the space of all real functions of class C^r on M. In addition to the obvious algebraic structure of $C^r(M)$, we shall use a normed algebra structure, the norm being obtained by introducing a Riemannian metric on M. The main results (Theorems 1 and 3) will be stated and proved in this section, using lemmas on differentiable manifolds which will be proved in the following section. Theorem 1 is straightforward, Theorem 3 is more difficult.

THEOREM 1. *If M is a compact differentiable manifold of class C^r, then $C^r(M)$ as an algebra determines the C^r structure of M; i.e., if $C^r(M)$ is isomorphic to $C^r(N)$, where M and N are compact differentiable manifolds of class C^r, then there is a differentiable homeomorphism of class C^r of M onto N with an inverse of class C^r.*

PROOF. $C^r(M)$ and $C^r(N)$ are inverse-closed (as subspaces of the spaces of all continuous functions on M and N respectively). Also, by Lemma 1, they are separating. It follows[2] that the space X of maximal ideals of $C^r(M)$ under the standard weak topology is homeomorphic to M, and the space Y of maximal ideals of $C^r(N)$ is homeomorphic to N. But there is a homeomorphism of X onto Y because of the assumed isomorphism $I(C^r(M)) = C^r(N)$, and hence a homeomorphism $H(M) = N$; furthermore, if $f \in C^r(M)$ and $F \in C^r(N)$, then $(I(f))(y) = f(H^{-1}(y))$ and $(I^{-1}(F))(x) = F(H(x))$. Therefore, by Lemma 2, H and H^{-1} are differentiable homeomorphisms of class C^r.

THEOREM 2. *Let M be a compact differentiable manifold of class C^r, provided with a Riemannian metric tensor g_{ij} of class C^{r-1}. For $f \in C^r(M)$ define*

$$\|f\| = \max_{x \in M} |f(x)| + \max_{x \in M} \left(g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \right)^{1/2}.[3]$$

Then $C^r(M)$ becomes a real, commutative, semi-simple, normed algebra with unit; if $r = 1$, $C^r(M)$ is a Banach algebra.

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[2] From the Gelfand theory. See, for example, Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, 1953, p. 55.

[3] The summation convention of tensor analysis is used throughout.

PROOF. As is well known, at each point $x \in M$,

$$\left(g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}\right)^{1/2} = \max \left| \frac{\partial f}{\partial x^i} \eta^i \right|,$$

the max being taken over the unit contravariant vectors η at x. It easily follows that $\|f_1 + f_2\| \leqq \|f_1\| + \|f_2\|$. Also,

$$\begin{aligned} \|f_1 f_2\| &= \max |f_1(x)| |f_2(x)| + \max_{x \in M, \eta \text{ at } x} \left| \left(f_1 \frac{\partial f_2}{\partial x^i} + f_2 \frac{\partial f_1}{\partial x^i}\right) \eta^i \right| \\ &\leqq \max |f_1(x)| \max |f_2(x)| \\ &\qquad + \max |f_1(x)| \max \left| \frac{\partial f_2}{\partial x^i} \eta^i \right| + \max |f_2(x)| \max \left| \frac{\partial f_1}{\partial x^i} \eta^i \right| \\ &\leqq \|f_1\| \|f_2\|. \end{aligned}$$

As for the completeness of $C^1(M)$, let f^α be a Cauchy sequence in $C^1(M)$. Then $f^\alpha(x)$ converges uniformly to a continuous function $f(x)$, while $(\partial f^\alpha / \partial x^i)\eta^i$ as a sequence of continuous functions on the tangent bundle $\overline{M}$ of unit contravariant vectors of M converges uniformerly over $\overline{M}$ to a function $F(\eta)$ continuous on $\overline{M}$. If $x^i = x^i(s)$ is a C^1 curve on M, with unit tangent vector $\eta^i(s) = dx^i/ds$, then

$$\int_0^s \frac{\partial f^\alpha}{\partial x^i} \eta^i(s) ds = f^\alpha(x(s)) - f^\alpha(x(0))$$

converges to $\int_0^s F(\eta(s))ds$. Hence

$$f(x(s)) - f(x(0)) = \int_0^s F(\eta(s))ds$$

so that $df(x(s))/ds = F(\eta(s))$ for every curve in M. Hence $f(x)$ is of class C^1, $(\partial f/\partial x^i)\eta^i = F(\eta)$ for all $\eta \in \overline{M}$, and $(\partial f^\alpha/\partial x^i)\eta^i$ converges uniformly over $\overline{M}$ to $(\partial f/\partial x^i)\eta^i$. Thus $C^1(M)$ is complete.

THEOREM 3. *Let M be a compact differentiable manifold of class C^r, provided with a Riemannian tensor g_{ij} of class $r-1$. Then $C^r(M)$ as a normed algebra (under the norm of Theorem 2) determines the Riemannian structure of M. More precisely, if as normed algebras $C^r(M)$ and $C^r(N)$ (M and N compact) are equivalent, there is an isometry of class C^r of M onto N.*

PROOF. According to Theorem 1, there is a nonsingular C^r-homeomorphism $H(M) = N$, which induces the equivalence $I(C^r(M)) = C^r(N)$. Let $y^i = y^i(x)$ be local equations of H, and let $I(f) = F$; then

$(\partial f/\partial x^i)(x) = (\partial F/\partial y^j)(y)(\partial y^j/\partial x^i)(x)$. If η is a contravariant vector at (x) and ζ is the vector $(\partial y/\partial x^i)\eta^i$ at $y(x)$, then $(\partial f/\partial x^i)\eta^i = (\partial F/\partial y^j)\zeta^j$. Regarding η and ζ as continuous linear functionals over $C^r(M)$ and $C^r(N)$ respectively according to the formulas $\eta(f) = (\partial f/\partial x^i)\eta^i$, $\zeta(F) = (\partial F/\partial y^j)\zeta^j$, then $\eta(f) = \zeta(F)$. Thus if I^* is the equivalence of the conjugate spaces of $C^r(M)$ and $C^r(N)$ induced by I, $I^*(\eta) = \zeta$. Hence $\|\eta\| = \|\zeta\|$. We now show $\|\eta\| = (g_{ij}\eta^i\eta^j)^{1/2}$, i.e. $\|\eta\|$ equals the magnitude of η. Without loss of generality we assume η is a unit vector. Now

$$\|\eta\| = \sup_f \frac{|\eta(f)|}{\|f\|} = \sup \frac{|\eta(f)|}{\max_{x\in M} |f| + \max_{\eta\in \overline{M}} |\eta(f)|} \leqq 1.$$

According to Lemma 3, given any ϵ, $\delta > 0$ there exists $\bar{f} \in C^r(M)$ such that

$$\eta(\bar{f}) = 1, \qquad \max_{x\in M} \left(g^{ij} \frac{\partial \bar{f}}{\partial x^i} \frac{\partial \bar{f}}{\partial x^j} \right)^{1/2} < 1 + \epsilon, \qquad \max_{x\in M} |\bar{f}(x)| < \delta.$$

Therefore $1 \leqq \|\bar{f}\| < 1 + \epsilon + \delta$, so $\|\eta\| \geqq 1$.

Thus we have shown $\|\eta\| = 1 = (g_{ij}\eta^i\eta^j)^{1/2}$. Similarly $\|\zeta\| = (h_{ij}\zeta^i\zeta^j)^{1/2}$, where h_{ij} is the metric tensor on N. It follows that $g_{ij}\eta^i\eta^j = h_{ij}\zeta^i\zeta^j$, so that $g_{ij}\eta^i\eta^j = h_{kl}(\partial y^k/\partial x^i)(\partial y^l \partial x^j)\eta^i\eta^j$ for arbitrary tangent vector η to M. Hence $g_{ij} = h_{kl}(\partial y^k/\partial x^i)(\partial y^l/\partial x^j)$, so that H is an isometry of class C^r.

2. Lemmas on differentiable manifolds.

Lemma 1. *If M is a differentiable manifold of class C^r, and if $P_0 \in M$ and K is a closed set not containing P_0, then there is an $f \in C^r(M)$ such that $f(P_0) = 1$, $f(P) = 0$ for $P \in K$.*

Proof. If (x) is an admissible coordinate system about P_0 with P_0 as origin, with $\sum x^i x^i < 4\delta^2$, and with range in the complement of K, then the following function f is the required function:

$$f = \left(1 - \frac{\sum x^i x^i}{\delta^2}\right)^{r+1} \qquad \text{for } \sum x^i x^i \leqq \delta^2,$$

$$f = 0 \qquad \text{for all other points of } M.$$

Lemma 2. *If H is a homeomorphism of a differentiable manifold M of class C^r onto a differentiable manifold N of class C^r, and if for every $F \in C^r(N)$ the induced function f defined by $f(x) = F(H(x))$ belongs to $C^r(M)$, then H is a differentiable homeomorphism of class C^r.*

Proof. Let $P \in M$, and let (x) and (y) be admissible coordinate systems about P and $H(P)$ respectively, the domain of (y) being $\sum y^i y^i < 3$. Let $y^i = y^i(x)$ be local equations of H. Define

$$A(\theta) = \frac{\int_1^\theta [(t-1)(t-2)]^r dt}{\int_1^2 [(t-1)(t-2)]^r dt}.$$

Then for each i the function F^i defined as follows is of class C^r over N:

$$\begin{aligned} F^i &= y^i && \text{for } \sum y^j y^j \leqq 1, \\ F^i &= y^i - y^i A(\sum y^j y^j) && \text{for } 1 \leqq \sum y^j y^j \leqq 2, \\ F^i &= 0 && \text{elsewhere on } N. \end{aligned}$$

The induced function $F^i(H(x))$ is by hypothesis of class C^r on M, hence $y^i(x)$ is of class C^r near P.

Lemma 3. *Let M be a differentiable manifold of class C^r provided with a Riemannian metric g_{ij} of class C^{r-1}. Let $P_0 \in M$, let η be a unit contravariant vector at P_0, and let ϵ, $e > 0$. Then there exists a function f of class C^r on M with the following properties:*

$$\frac{\partial f}{\partial x^i}\eta^i = 1 \qquad \text{at } P_0, \tag{1}$$

$$\sup_{x \in M}\left(g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\right)^{1/2} < 1 + \epsilon, \tag{2}$$

$$\sup_{x \in M} |f(x)| < e. \tag{3}$$

Proof. Let (x) be an admissible coordinate system about P_0, with P_0 as origin, with domain $\sum x^i x^i < 1$, with $g_{ij}(x) = \delta_{ij}$ at $(x) = (0)$, and such that the components of η in the coordinate system (x) are $(1, 0, \cdots, 0)$. Let $d < e$ be so small that by continuity of g^{ij} we have $|g^{ij}(x) - \delta^{ij}| < \epsilon/n^2$ for $0 \leqq \sum x^i x^i \leqq d^2$. Then the following is the required function:

$$\begin{aligned} f &= \left(1 - \frac{\sum x^i x^i}{d^2}\right)^{r+1} x^1 && \text{for } \sum x^i x^i \leqq d^2, \\ f &= 0 && \text{elsewhere on } M. \end{aligned}$$

To show this, note first that on $\sum x^i x^i = d^2$ all derivatives of f up to and including those of rth order are zero, so that f is of class C^r over M. Next, note that

$$\frac{\partial f}{\partial x^1} = \left(1 - \frac{\sum x^i x^i}{d^2}\right)^r \left(1 - \frac{\sum x^i x^i}{d^2} - \frac{2(r+1)(x^1)^2}{d^2}\right),$$

$$\frac{\partial f}{\partial x^\alpha} = -\frac{2(r+1)x^1 x^\alpha}{d^2}\left(1 - \frac{\sum x^i x^i}{d^2}\right)^r, \qquad \alpha = 2, 3, \cdots, n.$$

It is clear that at P_0 we have

$$\frac{\partial f}{\partial x^1} = 1, \qquad \frac{\partial f}{\partial x^\alpha} = 0, \qquad \frac{\partial f}{\partial x^i}\eta^i = 1.$$

Now compute as follows:

$$\begin{aligned}\sum \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^i} &= \left(1 - \frac{\sum x^i x^i}{d^2}\right)^{2r}\left[\left(1 - \frac{\sum x^i x^i}{d^2} - \frac{2(r+1)(x^1)^2}{d^2}\right)^2 \right.\\ &\qquad \left. + \frac{4(r+1)^2(x^1)^2}{d^4}\sum x^\alpha x^\alpha\right]\\ &= \left(1 - \frac{\sum x^i x^i}{d^2}\right)^{2r}\left[\left(1 - \frac{\sum x^i x^i}{d^2}\right)^2\right.\\ &\qquad \left. + \frac{4(r+1)(x^1)^2}{d^4}\left((2+r)\sum x^i x^i - d^2\right)\right].\end{aligned}$$

When $0 \leqq \sum x^i x^i \leqq d^2/(2+r)$, we have

$$\sum \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^i} \leqq \left(1 - \frac{x^i x^i}{d^2}\right)^{2r+2} \leqq 1$$

while when $d^2/(2+r) \leqq \sum x^i x^i \leqq d^2$, we have

$$\begin{aligned}\sum \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^i} &\leqq \left(1 - \frac{\sum x^i x^i}{d^2}\right)^{2r}\left[\left(1 - \frac{\sum x^i x^i}{d^2}\right)^2\right.\\ &\qquad \left. + \frac{4(r+1)\sum x^i x^i}{d^4}\left((2+r)\sum x^i x^i - d^2\right)\right]\\ &= \left(1 - \frac{\sum x^i x^i}{d^2}\right)^{2r}\left[1 - \frac{\sum x^i x^i}{d^2}(3+2r)\right]^2\\ &= (1-A)^{2r}(1-AB)^2\end{aligned}$$

where $A = (\sum x^i x^i)/d^2$, $B = 3+2r$. This non-negative quantity is zero when $A=1$, and equal to $((1+r)/(2+r))^{2r+2}$ when $A = 1/(2+r)$. Hence if we find its derivative with respect to A is zero at a unique value $\bar{A}$ between $A = 1/(2+r)$ and $A=1$, this locates its maximum either at $\bar{A}$ or at $A=1/(1+2r)$. But

$$\frac{d}{dA}((1-A)^{2r}(1-AB)^2)$$
$$= 2(1-A)^{2r-1}(1-AB)(-B+AB-r+rAB)$$

and since $1-AB<0$ for $1/(2+r)<A<1$, the only root in this A-interval is $A=(B+r)/(B+rB)$. For this value of A, we find

$$(1-A)^{2r}(1-AB)^2 = 4/(1+3/2r)^{2r} < 1.$$

Thus

$$\sum \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^i} \leqq 1 \qquad \text{for } 0 \leqq \sum x^i x^i \leqq d^2.$$

Now by our choice of d

$$g^{ij}(x) < \delta^{ij} + \epsilon/n^2 \qquad \text{for } 0 \leqq \sum x^i x^i \leqq d^2$$

so that

$$g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} < \sum \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^i} + \epsilon \leqq 1 + \epsilon.$$

Hence all over M

$$\left(g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\right)^{1/2} < 1 + \epsilon.$$

As for sup $|f(x)|$, note that $f=0$ at $(x)=(0)$ and on $\sum x^i x^i=d^2$, and that f is an odd function of x^1, so that max $|f(x)|$ in $0\leqq\sum x^i x^i \leqq d^2$ is the same as max $f(x)$ and occurs at a critical point of f not on $\sum x^i x^i=d^2$. From the form of $\partial f/\partial x^i$, it is seen that the critical point wanted is

$$x^\alpha = 0, \qquad \alpha = 2, 3, \cdots, n,$$
$$x^1 = d/(1+2r)^{1/2}.$$

Hence

$$\max |f(x)| = \left(1 - \frac{1}{1+2r}\right)^r \frac{d}{(1+2r)^{1/2}}$$
$$= \left(\frac{2r}{1+2r}\right)^r \frac{d}{(1+2r)^{1/2}} < d < e.$$

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