ON THE INTERSECTIONS OF THE COMPONENTS OF A DIFFERENCE POLYNOMIAL

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The purpose of this note is to prove the following theorem:

Solutions common to two distinct components1 of the manifold of a difference polynomial annul the separants of the polynomial.

We begin by considering a field $\mathbb{K}$, not necessarily a difference field, and a set of polynomials $F_1, F_2, \ldots, F_p$ in $\mathbb{K}[u_1, \ldots, u_q; x_1, \ldots, x_p]$, the $u_i$ and $x_j$ being indeterminates, where for each $j, i = 1, \ldots, p - 1$, $F_i$ is free of the $x_k, k > j$. We shall show that any zero of $F_1, \ldots, F_p$ which annuls no formal partial derivative $\partial F_j/\partial x_j$ belongs to just one component of $\{F_1, \ldots, F_p\}$. Furthermore, this component is of dimension $q$.

Proof. Let $u_i = y_i, i = 1, \ldots, q; x_j = \alpha_j, j = 1, \ldots, p,$ be a zero of $F_1, \ldots, F_p$ which annuls no $\partial F_j/\partial x_j$. If $\gamma'_1, \ldots, \gamma'_q; \alpha'_1, \ldots, \alpha'_p$ is a zero of $F_1, \ldots, F_p$ which specializes to $\gamma_1, \ldots, \gamma_q; \alpha_1, \ldots, \alpha_p$, then this zero too annuls no $\partial F_j/\partial x_j$. It follows from this that $\alpha'_1$ is algebraic over $\mathbb{K}(\gamma'_1, \ldots, \gamma'_q)$, and that for each $k, 1 < k \leq p, \alpha'_k$ is algebraic over $\mathbb{K}(\gamma'_1, \ldots, \gamma'_q; \alpha'_1, \ldots, \alpha'_{k-1})$. This implies that a component of the manifold of $\{F_1, \ldots, F_p\}$ containing $\gamma_1, \ldots, \gamma_q; \alpha_1, \ldots, \alpha_p$ is of dimension at most $q$.

We let $u_i = t_i + \gamma_i, i = 1, \ldots, q; x_j = \alpha_j + h_j, j = 1, \ldots, p$. Here the $t_i$ denote new indeterminates and the $h_j$ certain formal series in positive integral powers of the $t_i$. We shall show that the $h_j$ may be so chosen that these substitutions annul $F_1, \ldots, F_p$. In fact, the lemma proved in [3] shows that for each $k, 1 \leq k \leq p, \gamma'_k$ is algebraic over $\mathbb{K}(\gamma'_1, \ldots, \gamma'_q)$, and that for each $k, 1 < k \leq p, \gamma'_k$ is algebraic over $\mathbb{K}(\gamma'_1, \ldots, \gamma'_q; \gamma'_1, \ldots, \gamma'_{k-1})$. This implies that a component of the manifold of $\{F_1, \ldots, F_p\}$ containing $\gamma_1, \ldots, \gamma_q; \alpha_1, \ldots, \alpha_p$ is of dimension at most $q$.

With the $h_j$ as described let $\Sigma$ denote the set of polynomials in $\mathbb{K}[u_1, \ldots, u_q; x_1, \ldots, x_p]$ which are annulled by the above substitutions. Evidently $\Sigma$ is a prime $p$. i. (polynomial ideal). Its dimen-

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1 The term "component," not previously defined for difference manifolds, is to have the expected meaning: a component is a maximal irreducible submanifold of a manifold. For definitions of other terms and symbols see [2; 3; 4].

2 As in Chapter IV of [1] this notation indicates the perfect polynomial ideal generated by $F_1, F_2, \ldots, F_p$. 

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sion is \( q \) and the \( u_i \) form a parametric set. For evidently \( \Sigma \) can contain no polynomial in the \( u_i \) alone, while the conclusion of the preceding paragraph but one shows that its dimension cannot exceed \( q \). The result of that paragraph also shows that no component of \( \{ F_1, \cdots, F_p \} \) can properly contain the manifold of \( \Sigma \), for then its dimension would exceed \( q \). Hence this manifold is itself a component of \( \{ F_1, \cdots, F_p \} \).

Let \( M \) be a component of \( \{ F_1, \cdots, F_p \} \) which contains \( y_1, \cdots, y_q; \alpha_1, \cdots, \alpha_p \), and let \( \Lambda \) be the prime p. i. in \( K[u_1, \cdots, u_q; x_1, \cdots, x_p] \) whose manifold is \( M \). We must show that \( \Lambda \) is \( \Sigma \). If \( \Lambda \) is of dimension 0 then, because \( \Sigma \) vanishes for a zero of \( \Lambda \), and every zero must be a generic zero, \( \Sigma \) is contained in \( \Lambda \). Since the manifolds of both are components of the same manifold, it follows that \( \Lambda = \Sigma \) (and that \( q = 0 \)). We suppose that \( \Lambda \) is of positive dimension, and that \( \Lambda \) and \( \Sigma \) are distinct. Then, since \( \Lambda \) cannot contain \( \Sigma \), there is a polynomial \( P \) in \( \Sigma \) which is not in \( \Lambda \). Then \( \Lambda \) possesses a zero not annulling \( P \) of the form

\[ u_i = y_i + g_i, \quad i = 1, \cdots, q; \]
\[ x_j = \alpha_j + f_j, \quad j = 1, \cdots, p, \]

where the \( g_i \) and the \( f_j \) are series in positive integral powers of a parameter \( t \).

It is evident that (1) is a zero of \( F_1, \cdots, F_p \). We may also obtain a zero of these polynomials of the form

\[ u_i = y_i + g_i, \quad i = 1, \cdots, q; \]
\[ x_j = \alpha_j + f'_j, \quad j = 1, \cdots, p, \]

where the \( f'_j \) are again series in positive integral powers of \( t \), and each \( f'_j \) is obtained by replacing the \( t_i, i = 1, \cdots, p \), in \( h_j \) by the corresponding \( g_i \). It is evident from the manner of formation of (2) that it is a zero of \( \Sigma \).

We replace the \( u_i \) in \( F_1 \) by \( y_i + g_i, i = 1, \cdots, q \). There results a polynomial \( \overline{F}_1 \) in \( x_1 \) with coefficients power series in \( t \). \( \overline{F}_1 \) vanishes, but its formal derivative \( d\overline{F}_1/dx_1 \) does not, when we put \( t = 0, x_1 = \alpha_1 \). It follows that there is a unique series \( f''_1 \) in positive integral powers of \( t \) such that \( x_1 = \alpha_1 + f''_1 \) is a solution of \( \overline{F}_1 = 0 \). We now replace the \( u_i, i = 1, \cdots, q \), and \( x_1 \) in \( F_2 \) by \( y_i + g_i \) and \( \alpha_1 + f''_1 \) respectively to obtain a polynomial \( \overline{F}_2 \) in \( x_2 \) with coefficients power series in \( t \). As before, we see that \( \overline{F}_2 = 0 \) possesses a solution \( x_2 = \alpha_2 + f''_2 \), where \( f''_2 \) is a series in positive integral powers of \( t \). This series is unique. Continuing in this way we find uniquely determined \( f''_j, j = 1, \cdots, p \), which are series in positive integral powers of \( t \) such that \( u_i = y_i + g_i, \)
\( i = 1, \ldots, q; x_j = \alpha_j + f_j', j = 1, \ldots, p, \) is a zero of \( F_1, \ldots, F_p. \)

The uniqueness of the \( f_j' \) shows that (1) and (2) are identical. Hence (1) annuls \( \Sigma, \) and, in particular, it annuls \( P. \) We have thus obtained a contradiction. This completes the proof of our statement concerning the zeros of \( F_1, \ldots, F_p. \)

Now let \( \mathcal{J} \) be a difference field and \( A \) a polynomial of \( \mathcal{J} \{ y_1, \ldots, y_n \}. \) We shall prove the theorem stated at the beginning of this note. We may suppose that a transform of some \( y_i, \) say of \( y_n, \) appears effectively in \( A. \) Let \( y_i = \alpha_i, i = 1, \ldots, n, \) be a zero of \( A. \) It will suffice to assume that the \( \alpha_i \) are not a zero of the \( y_n \)-separant of \( A \) and show that this implies that only one component of the manifold of \( A \) contains the \( \alpha_i. \)

It is evident that the \( \alpha_i \) must annul just one irreducible factor, say \( F, \) of \( A, \) and do not annul the \( y_n \)-separant of \( F. \) Hence we need merely show that the \( \alpha_i \) are contained in only one component of the manifold of \( F. \) We shall suppose that this is not so and obtain a contradiction. We assume first that \( F \) is of equal order and effective order in \( y_n. \)

Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) denote two distinct components of the manifold of \( F, \) each containing the \( \alpha_i. \) Let \( \Sigma_1 \) and \( \Sigma_2 \) denote the corresponding reflexive prime difference ideals. We denote by \( h \) the order of \( F \) in \( y_n. \) Since the \( \alpha_i \) do not annul the \( y_n \)-separant of \( F, y_1, \ldots, y_{n-1} \) constitute a parametric set for both \( \Sigma_1 \) and \( \Sigma_2, \) and these ideals are both of order \( h \) in \( y_n. \)

We choose an integer \( m \) such that the first \( m+1 \) polynomials of a characteristic sequence of \( \Sigma_1 \) do not constitute the beginning of a characteristic sequence of \( \Sigma_2. \) Let \( \Sigma_{1m} \) and \( \Sigma_{2m} \) denote the sets consisting of those polynomials of \( \Sigma_1 \) and \( \Sigma_2 \) respectively which involve the \( y_n^k, 0 \leq k \leq m+h, \) and a finite subset \( S \) of the \( y_{ij}, i < n. \) \( S \) is to include all those \( y_{ij}, i < n, \) which appear effectively, or whose transforms appear effectively, in \( F, F_1, \ldots, F_m \) or in the first \( m+1 \) polynomials of a characteristic sequence of \( \Sigma_1 \) or in the first \( m+1 \) polynomials of a characteristic sequence of \( \Sigma_2. \)

\( \Sigma_{1m} \) and \( \Sigma_{2m} \) may be regarded as prime \( \mathfrak{p}.i.'s \) in the ring \( \mathcal{J}[S, y_{n0}, y_{n1}, \ldots, y_{nm+h}]. \) The \( y_{ij} \) of \( S \) and the \( y_{nk}, k < h, \) constitute a parametric set for both \( \Sigma_{1m} \) and \( \Sigma_{2m}. \) Let \( s \) denote the number of indeterminates in this parametric set.

Our earlier result concerning polynomial ideals shows that there is a unique component \( \mathcal{M} \) of the manifold of \( \{ F, F_1, \ldots, F_m \} \) regarded as an ideal of \( \mathcal{J}[S, y_{n0}, y_{n1}, \ldots, y_{nm+h}]. \) which contains the zero \( y_{ij} = \alpha_{ij} \) of this ideal. The dimension of \( \mathcal{M} \) is \( s, \) for \( s \) corresponds to \( q \) of the earlier proof.

Now both \( \Sigma_{1m} \) and \( \Sigma_{2m} \) contain \( \{ F, F_1, \ldots, F_m \} \), while both have
the zero $y_{ij} = \alpha_{ij}$. Hence their manifolds are in $\mathcal{M}$. Since their manifolds are of dimension $s$, however, they must coincide with $\mathcal{M}$. Hence $\Sigma_{1m}$ and $\Sigma_{2m}$ are identical. But $m$ was chosen so that $\Sigma_{1m}$ contains a polynomial which is not in $\Sigma_{2m}$, namely one of the first $m + 1$ polynomials of a characteristic sequence of $\Sigma_1$. We have obtained a contradiction. This completes the proof of the theorem in the case that $F$ is of equal order and effective order in $y_n$.

If the order of $F$ in $y_n$ exceeds its effective order by $d > 0$, we replace each $y_{nk}$ in $F$ by $z_{k-d}$, where $z$ is a new indeterminate, and subscripts attached to $z$ denote transforming. $F$ goes into an irreducible polynomial $\overline{F}$ which is of equal order and effective order in $z$.

Evidently each component $\mathcal{M}$ of the manifold of $\overline{F}$ corresponds to a unique component $\mathcal{M}$ of the manifold of $F$, and, conversely, each component of the manifold of $F$ is obtained from a unique component of the manifold of $\overline{F}$. The correspondence may be described as follows: each solution in $\mathcal{M}$ is obtained from a solution in $\overline{M}$ by leaving unchanged the elements assigned as values to $y_1, \ldots, y_{n-1}$, and assigning to $y_n$ an element whose $d$th transform is the element assigned as the value of $z$ in $\overline{M}$. This correspondence carries solutions common to two components of the manifold of $F$ into solutions common to two components of the manifold of $\overline{F}$. Solutions annulling the $y_n$-separant of $F$ correspond to solutions annulling the $z$-separant of $\overline{F}$.

The preceding proof shows that the theorem stated at the beginning of this note holds for $\overline{F}$. The correspondence just described shows that its truth for $\overline{F}$ implies its truth for $F$. Hence it is true in general.

References


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