ON THE INTERSECTIONS OF THE COMPONENTS OF
A DIFFERENCE POLYNOMIAL

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The purpose of this note is to prove the following theorem:

Solutions common to two distinct components\(^1\) of the manifold of a
difference polynomial annul the separants of the polynomial.

We begin by considering a field \(K\), not necessarily a difference
field, and a set of polynomials \(F_1, F_2, \ldots, F_p\) in \(K[u_1, \ldots, u_q; x_1, \ldots, x_p]\), the \(u_i\) and \(x_j\) being indeterminates, where for each
\(j, j=1, \ldots, p-1, F_j\) is free of the \(x_k, k>j\). We shall show that any
zero of \(F_1, \ldots, F_p\) which annuls no formal partial derivative \(\partial F_j/\partial x_j\)
belongs to just one component of \(\{F_1, \ldots, F_p\}_0\).\(^2\) Furthermore, this
component is of dimension \(q\).

Proof. Let \(u_i=y_i, i=1, \ldots, q; x_j=\alpha_j, j=1, \ldots, p,\) be a zero
of \(F_1, \ldots, F_p\) which annuls no \(\partial F_j/\partial x_j\). If \(\gamma'_1, \ldots, \gamma'_q; \alpha'_1, \ldots, \alpha'_p\)
is a zero of \(F_1, \ldots, F_p\) which specializes to \(\gamma_1, \ldots, \gamma_q; \alpha_1, \ldots, \alpha_p,\)
then this zero too annuls no \(\partial F_j/\partial x_j\). It follows from this that \(\alpha'_1\) is
algebraic over \(K(\gamma'_1, \ldots, \gamma'_q)\), and that for each \(k, 1<k\leq p, \alpha'_k\) is
algebraic over \(K(\gamma'_1, \ldots, \gamma'_q; \alpha'_1, \ldots, \alpha'_{k-1})\). This implies that a
component of the manifold of \(\{F_1, \ldots, F_p\}_0\) containing \(\gamma_1, \ldots, \gamma_q;\)
\(\alpha_1, \ldots, \alpha_p\) is of dimension at most \(q\).

We let \(u_i=t_i+\gamma_i, i=1, \ldots, q; x_j=\alpha_j+h_j, j=1, \ldots, p.\) Here the
t\(_i\) denote new indeterminates and the \(h_j\) certain formal series in positive
integral powers of the \(t\). We shall show that these substitutions annul \(F_1, \ldots, F_p\). In fact, the
lemma proved in \([3]\) shows that for each \(k, 1\leq k\leq p,\) we may annul
\(F_k\) by substitutions \(u_i=t_i+\gamma_i, i=1, \ldots, p, x_j=s_j+\alpha_j, j<k, x_k=\alpha_k+
h_k',\) where the \(s_j, j=1, \ldots, p,\) are new indeterminates, and \(h_k'\) is a
formal series in positive integral powers of the \(t_i\) and \(s_j, j<k.\) For
\(h_1\) we take \(h'_1;\) for \(h_2\) we take the result of replacing \(s_1\) in \(h'_2\) by \(h'_1,\) and
so on.

With the \(h_j\) as described let \(\Sigma\) denote the set of polynomials in
\(K[u_1, \ldots, u_q; x_1, \ldots, x_p]\) which are annulled by the above sub-
tstitutions. Evidently \(\Sigma\) is a prime p. i. (polynomial ideal). Its dimen-

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\(^1\) The term "component," not previously defined for difference manifolds, is to
have the expected meaning: a component is a maximal irreducible submanifold of a
manifold. For definitions of other terms and symbols see \([2; 3; 4]\).

\(^2\) As in Chapter IV of \([1]\) this notation indicates the perfect polynomial ideal
generated by \(F_1, F_2, \ldots, F_p\).
sion is $q$ and the $u_i$ form a parametric set. For evidently $\Sigma$ can contain no polynomial in the $u_i$ alone, while the conclusion of the preceding paragraph but one shows that its dimension cannot exceed $q$. The result of that paragraph also shows that no component of $\{F_1, \ldots, F_p\}_0$ can properly contain the manifold of $\Sigma$, for then its dimension would exceed $q$. Hence this manifold is itself a component of $\{F_1, \ldots, F_p\}_0$.

Let $\mathcal{M}$ be a component of $\{F_1, \ldots, F_p\}_0$ which contains $\gamma_1, \ldots, \gamma_q; \alpha_1, \ldots, \alpha_p$, and let $\Lambda$ be the prime p. i. in $K[u_1, \ldots, u_q; x_1, \ldots, x_p]$ whose manifold is $\mathcal{M}$. We must show that $\Lambda$ is $\Sigma$. If $\Lambda$ is of dimension 0 then, because $\Sigma$ vanishes for a zero of $\Lambda$, and every zero must be a generic zero, $\Sigma$ is contained in $\Lambda$. Since the manifolds of both are components of the same manifold, it follows that $\Lambda = \Sigma$ (and that $q = 0$). We suppose that $\Lambda$ is of positive dimension, and that $\Lambda$ and $\Sigma$ are distinct. Then, since $\Lambda$ cannot contain $\Sigma$, there is a polynomial $P$ in $\Sigma$ which is not in $\Lambda$. Then $\Lambda$ possesses a zero not annulling $P$ of the form

$$
\begin{align*}
\gamma_i &= \gamma_i + g_i, & i &= 1, \ldots, q; \\
x_j &= \alpha_j + f_j, & j &= 1, \ldots, p,
\end{align*}
$$

where the $g_i$ and the $f_j$ are series in positive integral powers of a parameter $t$.

It is evident that (1) is a zero of $F_1, \ldots, F_p$. We may also obtain a zero of these polynomials of the form

$$
\begin{align*}
\gamma_i &= \gamma_i + g_i, & i &= 1, \ldots, q; \\
x_j &= \alpha_j + f_j', & j &= 1, \ldots, p,
\end{align*}
$$

where the $f_j'$ are again series in positive integral powers of $t$, and each $f_j'$ is obtained by replacing the $t_i, i = 1, \ldots, p$, in $h_j$ by the corresponding $g_i$. It is evident from the manner of formation of (2) that it is a zero of $\Sigma$.

We replace the $u_i$ in $F_1$ by $\gamma_i + g_i, i = 1, \ldots, q$. There results a polynomial $\bar{F}_1$ in $x_1$ with coefficients power series in $t$. $\bar{F}_1$ vanishes, but its formal derivative $d\bar{F}_1/dx_1$ does not, when we put $t = 0$, $x_1 = \alpha_1$. It follows that there is a unique series $\bar{f}'_1$ in positive integral powers of $t$ such that $x_1 = \alpha_1 + \bar{f}'_1$ is a solution of $\bar{F}_1 = 0$. We now replace the $u_i, i = 1, \ldots, q$, and $x_1$ in $\bar{F}_2$ by $\gamma_i + g_i$ and $\alpha_1 + \bar{f}'_1$ respectively to obtain a polynomial $\bar{F}_2$ in $x_2$ with coefficients power series in $t$. As before, we see that $\bar{F}_2 = 0$ possesses a solution $x_2 = \alpha_2 + \bar{f}'_2$, where $\bar{f}'_2$ is a series in positive integral powers of $t$. This series is unique. Continuing in this way we find uniquely determined $f'_j, j = 1, \ldots, p$, which are series in positive integral powers of $t$ such that $u_i = \gamma_i + g_i$. 


\[ i = 1, \ldots, q; x_j = \alpha_j + f''_j, j = 1, \ldots, p, \] is a zero of \( F_1, \ldots, F_p. \)

The uniqueness of the \( f''_j \) shows that (1) and (2) are identical. Hence (1) annuls \( \Sigma \), and, in particular, it annuls \( P \). We have thus obtained a contradiction. This completes the proof of our statement concerning the zeros of \( F_1, \ldots, F_p. \)

Now let \( J \) be a difference field and \( A \) a polynomial of \( J \{ y_1, \ldots, y_n \} \). We shall prove the theorem stated at the beginning of this note. We may suppose that a transform of some \( y_i \), say of \( y_n \), appears effectively in \( A \). Let \( y_i = \alpha_i, i = 1, \ldots, n, \) be a zero of \( A \). It will suffice to assume that the \( \alpha_i \) are not a zero of the \( y_n \)-separant of \( A \) and show that this implies that only one component of the manifold of \( A \) contains the \( \alpha_i \).

It is evident that the \( \alpha_i \) must annul just one irreducible factor, say \( F \), of \( A \), and do not annul the \( y_n \)-separant of \( F \). Hence we need merely show that the \( \alpha_i \) are contained in only one component of the manifold of \( F \). We shall suppose that this is not so and obtain a contradiction. We assume first that \( F \) is of equal order and effective order in \( y_n \).

Let \( M_1 \) and \( M_2 \) denote two distinct components of the manifold of \( F \), each containing the \( \alpha_i \). Let \( \Sigma_1 \) and \( \Sigma_2 \) denote the corresponding reflexive prime difference ideals. We denote by \( h \) the order of \( F \) in \( y_n \). Since the \( \alpha_i \) do not annul the \( y_n \)-separant of \( F \), \( y_1, \ldots, y_{n-1} \) constitute a parametric set for both \( \Sigma_1 \) and \( \Sigma_2 \), and these ideals are both of order \( h \) in \( y_n \).

We choose an integer \( m \) such that the first \( m+1 \) polynomials of a characteristic sequence of \( \Sigma_1 \) do not constitute the beginning of a characteristic sequence of \( \Sigma_2 \). Let \( \Sigma_{1m} \) and \( \Sigma_{2m} \) denote the sets consisting of those polynomials of \( \Sigma_1 \) and \( \Sigma_2 \) respectively which involve the \( y_n^k, 0 \leq k \leq m+h \), and a finite subset \( S \) of the \( y_{ij}, i < n \). \( S \) is to include all those \( y_{ij}, i < n \), which appear effectively, or whose transforms appear effectively, in \( F, F_1, \ldots, F_m \) or in the first \( m+1 \) polynomials of a characteristic sequence of \( \Sigma_1 \) or in the first \( m+1 \) polynomials of a characteristic sequence of \( \Sigma_2 \).

\( \Sigma_{1m} \) and \( \Sigma_{2m} \) may be regarded as prime \( \mathbb{R} \)s in the ring \( J[S, y_{n0}, y_{n1}, \ldots, y_{n,m+h}] \). The \( y_{ij} \) of \( S \) and the \( y_{nk}, k < h \), constitute a parametric set for both \( \Sigma_{1m} \) and \( \Sigma_{2m} \). Let \( s \) denote the number of indeterminates in this parametric set.

Our earlier result concerning polynomial ideals shows that there is a unique component \( \mathcal{M} \) of the manifold of \( \{ F, F_1, \ldots, F_m \}_0 \), regarded as an ideal of \( J[S, y_{n0}, y_{n1}, \ldots, y_{n,m+h}] \), which contains the zero \( y_{ij} = \alpha_{ij} \) of this ideal. The dimension of \( \mathcal{M} \) is \( s \), for \( s \) corresponds to \( q \) of the earlier proof.

Now both \( \Sigma_{1m} \) and \( \Sigma_{2m} \) contain \( \{ F, F_1, \ldots, F_m \}_0 \), while both have
the zero \( y_{ij} = \alpha_{ij} \). Hence their manifolds are in \( \mathcal{M} \). Since their manifolds are of dimension \( s \), however, they must coincide with \( \mathcal{M} \). Hence \( \Sigma_{1m} \) and \( \Sigma_{2m} \) are identical. But \( m \) was chosen so that \( \Sigma_{1m} \) contains a polynomial which is not in \( \Sigma_{2m} \), namely one of the first \( m + 1 \) polynomials of a characteristic sequence of \( \Sigma_1 \). We have obtained a contradiction. This completes the proof of the theorem in the case that \( F \) is of equal order and effective order in \( y_n \).

If the order of \( F \) in \( y_n \) exceeds its effective order by \( d > 0 \), we replace each \( y_{nk} \) in \( F \) by \( z_{k-d} \), where \( z \) is a new indeterminate, and subscripts attached to \( z \) denote transforming. \( F \) goes into an irreducible polynomial \( \overline{F} \) which is of equal order and effective order in \( z \).

Evidently each component \( \overline{\mathcal{M}} \) of the manifold of \( \overline{F} \) corresponds to a unique component \( \mathcal{M} \) of the manifold of \( F \), and, conversely, each component of the manifold of \( F \) is obtained from a unique component of the manifold of \( \overline{F} \). The correspondence may be described as follows: each solution in \( \overline{\mathcal{M}} \) is obtained from a solution in \( \mathcal{M} \) by leaving unchanged the elements assigned as values to \( y_1, \ldots, y_{n-1} \), and assigning to \( y_n \) an element whose \( d \)th transform is the element assigned as the value of \( z \) in \( \overline{\mathcal{M}} \). This correspondence carries solutions common to two components of the manifold of \( F \) into solutions common to two components of the manifold of \( \overline{F} \). Solutions annulling the \( y_n \)-separant of \( F \) correspond to solutions annulling the \( z \)-separant of \( \overline{F} \).

The preceding proof shows that the theorem stated at the beginning of this note holds for \( \overline{F} \). The correspondence just described shows that its truth for \( \overline{F} \) implies its truth for \( F \). Hence it is true in general.

References


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