PARTIALLY CONTINUOUS DECOMPOSITIONS

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1. Introduction. In this paper we consider collections G of mutually exclusive compact sets whose sum is a subset X of a metric space. (We use the convention that a *compact* set is closed.) We shall be interested in whether or not G is upper or lower semicontinuous. See [8; 9] for treatments of such collections.

The collection G is *upper semicontinuous* (or *lower semicontinuous*) if for each element g of G and each positive number ϵ there is a positive number δ such that if g' is an element of G such that $\rho(g, g') < \delta$, then $g' \subset V(g, \epsilon)$ (or $g \subset V(g', \epsilon)$). Here $\rho(A, B)$ denotes the distance between the sets A and B and $V(A, \epsilon)$ is the set of all points whose distance from A is less than ϵ.

If $\{g_i\}$ is a sequence of elements of G, the *superior limit* (or *inferior limit*) of $\{g_i\}$ is the set of all points p such that each neighborhood of p intersects infinitely many (or all but a finite number of the) elements of $\{g_i\}$. If these two limits are equal, they are called the limit of $\{g_i\}$, and $\{g_i\}$ is said to converge to this limit.

In the case where X is compact, the collection G is upper semicontinuous if for each converging sequence $\{p_i\}$ of points of X, the superior limit of $\{g_i\}$ (g_i contains p_i) lies in an element of G. The collection is lower semicontinuous if for each such sequence, each element that intersects the inferior limit of $\{g_i\}$ lies in it. The collection G is continuous if it is both upper and lower semicontinuous.

The following scheme reminds us of the definitions of upper and lower semicontinuity as given in real variables.

$$\text{Lower semicontinuity} \leftrightarrow g_0 \subset \text{limit } \{g_i\}.$$
$$\text{Upper semicontinuity} \leftrightarrow g_0 \supset \text{limit } \{g_i\}.$$
$$\text{Continuity} \leftrightarrow g_0 = \text{limit } \{g_i\}.$$

Here $\{g_i\}$ represents a converging sequence of elements of G and g_0 is an element of G that intersects the limit of this sequence.

If the elements of G are connected, G is called a *monotone* decomposition of X. An element (collection) is *nondegenerate* if it has more than one point (element). We show in Theorem 4 that if G is a lower semicontinuous collection each of whose elements is a nondegenerate continuous curve and X is a closed plane set, then G is also upper semicontinuous.

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2. **Decompositions and transformations.** While upper semicontinuous decompositions have been widely studied, the notion of lower semicontinuous decompositions has not been used nearly so widely. We list some relationships between partially continuous decompositions and transformations. For the most part, these statements are well known for upper semicontinuous decompositions and are no more difficult to prove in the lower semicontinuous case.

Suppose f is a transformation (not necessarily continuous) of a compact metric space X onto a set $f(X)$ such that for each point y of $f(X)$, $f^{-1}(y)$ is closed. Then f imposes a decomposition G on X—the elements of G are the inverses of the points of $f(X)$. Here are two tests for the upper or lower semicontinuity of G:

1. *G is upper semicontinuous if and only if, for each closed subset A of X, $f^{-1}f(A)$ is closed.*

2. *G is lower semicontinuous if and only if, for each open subset D of X, $f^{-1}f(D)$ is open.*

Now suppose $f(X)$ has a topology and f is continuous. Then we have the following results:

3. *G is upper semicontinuous if $f(X)$ is metric (even Hausdorff).*

4. *G is lower semicontinuous if $f(X)$ is T_1 and f is open.*

Conversely, suppose G is a decomposition of X. Then there is a continuous transformation g of X determining G and satisfying the following conditions.

5. *$g(X)$ is metric if G is upper semicontinuous.*

6. *$g(X)$ is T_1 and g is open if G is lower semicontinuous.*

We may regard the elements of G as the elements of the space $g(X)$. If U is an open set in X, the collection of elements of G which lie in U is an open set in the metric space $g(X)$ of statement 5; the collection of elements of X which intersect U is an open set in the T_1 space $g(X)$ of statement 6.

3. **Reducing lower semicontinuous collections.** Here we see how to reduce a locally compact continuum with a lower semicontinuous decomposition to a compact continuum with a decomposition which comes closer to being continuous. We define a *continuum* to be a closed connected set which is not necessarily compact.

If g_1, g_2 are two sets, the *Hausdorff distance* $H(g_1, g_2)$ between them is the least upper bound of $\rho(x, g_i)$; $i = 1, 2$; $x \in g_1 + g_2$.

THEOREM 1. *Suppose G is a monotone lower semicontinuous decomposition of a locally compact continuum X and G has more than one element. For each element g_0 of G and each positive number ϵ there is a compact continuum M such that M is the sum of elements of G, M prop-*

erly contains g_0, and the Hausdorff distance between each pair of elements of G in M is less than ϵ.

PROOF. Regard X as all of space. Since a locally compact connected metric space is perfectly separable, there is a countable basis for X— that is, a countable collection $\{D_i\}$ of open sets such that each open set in X is the sum of elements of $\{D_i\}$.

If there is a continuum in $X - D_1$ that properly contains g_0 and which is the sum of elements of G, let M_1 be such a continuum; otherwise $M_1 = X$. If there is a continuum in $M_1 - D_2$ which properly contains g_0 and is the sum of elements of G, let M_2 be such a continuum; otherwise $M_2 = M_1$. In general, M_{i+1} is a subcontinuum of M_i which properly contains g_0 and is the sum of elements of G; it misses D_{i+1} if this is possible; otherwise $M_{i+1} = M_i$.

The intersection of M_1, M_2, $\cdots$ is a closed set M' that contains g_0 and is the sum of elements of G. We show that $M' = g_0$.

Let $N(\epsilon)$ be the sum of all elements of G in M' whose distance (not Hausdorff distance) from g_0 is greater than or equal to ϵ. It follows from the lower semicontinuity of G that $N(\epsilon)$ is closed. Since $M' = g_0 + N(1) + N(1/2) + N(1/3) + \cdots$, it follows from the Baire category theorem that unless $g_0 = M'$, there is an integer j such that $N(1/j)$ contains a nonnull subset E of M' open relative to M'.

Assume g_0 is not a component of M'. Let R be a continuum in $M' - N(1/j)$ that properly contains g_0 and M'' be the closure of the sum of all elements of G that intersect R. Then M'' does not intersect E, but properly contains g_0, and is the sum of elements of G. If p is a point of E, there is an integer r such that D_r contains p and $D_r \cdot M'$ lies in E. But M_r would not contain p because M'' is a continuum in $M' - D_r$ (hence in $M_{r-1} - D_r$) which properly contains g_0 and is the sum of elements of G. Hence, the assumption that g_0 is not a component of M' has led to the contradiction that the point p of M' does not belong to M'.

Since g_0 is compact, lies in an open set D whose closure is compact, and is a component of the intersection of the decreasing sequence M_1, M_2, $\cdots$ of continua, g_0 is this intersection.

Since G is lower semicontinuous there is a positive number δ such that if g is an element of G such that $\rho(g, g_0) < \delta$, then for each point p of g_0, $\rho(p, g) < \epsilon/2$. Since g_0 is the intersection of the continua of M_1, M_2, $\cdots$ and lies in an open set whose closure is compact, there is an integer k such that M_k is compact and for each point p of M_k, $\rho(p, g_0)$ is less than either $\epsilon/2$ or δ. We can let $M = M_k$ because if g is an element of G in M_k, then $H(g, g_0) < \epsilon/2$.

We note that the preceding theorem is not true if we replace lower

semicontinuity by upper semicontinuity. The theorem implies that if g_0 is an element of G there is a sequence of different elements of G converging to g_0.

Consider two examples of lower semicontinuous decompositions.

EXAMPLE 1. We describe a lower semicontinuous decomposition G_ϵ of a unit cube into arcs and disks. The decomposition is not upper semicontinuous.

Consider a base B of this cube. If S is a slice of the cube parallel to B and at an irrational distance from B, S is an element of G_ϵ.

If S is a slice of the cube at a distance p/q (in lowest terms) from B, there is a continuous decomposition $G(S, \epsilon/q)$ of S such that each element of $G(S, \epsilon/q)$ is an arc which comes within ϵ/q of each point of S. Each element of $G(S, \epsilon/q)$ is an element of G.

EXAMPLE 2. We may be unable to get M for Theorem 1 on which G is continuous. We give here an example of a lower semicontinuous decomposition G of a Hilbert cube such that if G' is any infinite sub-collection of G whose sum is a continuum, G' fails to be upper semi-continuous.

We may regard the Hilbert cube as the cartesian product $C_1 \times C_2 \times C_3 \times \cdots$ where C_i is a unit cube. Frequently a Hilbert cube is considered as the cartesian product of closed intervals I_n where $I_n = [-1/n, 1/n]$ but by adjusting the metric we may suppose $I_n = [0, 1]$. Similarly, we may choose a metric for $C_1 \times C_2 \times C_3 \cdots$ to give it the familiar metric for the Hilbert cube. For each positive number ϵ and each cube C_i let $G(\epsilon, C_i)$ be a particular decomposition of C_i into disks and arcs as described in Example 1. There is a count-able collection $W(\epsilon, C_i) = \{w_j\}$ of arcs in $G(\epsilon, C_i)$ such that each continuum in C_i which is the sum of infinitely many elements of $G(\epsilon, C_i)$ contains some element of $W(\epsilon, C_i)$. Let $G'(\epsilon, C_i) = G(\epsilon, C_i) - W(\epsilon, C_i)$.

If $g^1 \in G'(1, C_1)$, $g^1 \times C_2 \times C_3 \times \cdots$ is an element of G. If $w_i^1 \in W(1, C_1)$, consider $G(\epsilon_i^1, C_2)$ where $\epsilon_i^1 = 1/iq$ and w_i^1 is in the slice of C_1 at a distance of p/q from the base of C_1.

If $g^2 \in G'(\epsilon_i^1, C_2)$, $w_i^1 \times g^2 \times C_3 \times \cdots$ is an element of G. If $w_j^2 \in W(\epsilon_i^1, C_2)$, consider $G(\epsilon_j^2, C_3)$ where $\epsilon_j^2 = \epsilon_i^1/jq$ and w_j^2 is in the slice of C_2 at a distance of p/q from the base of C_2. If $g^3 \in G'(\epsilon_j^2, C_3)$, $w_i^1 \times w_j^2 \times g^3 \times C_4 \times \cdots$ is an element of G.

The process is continued. An element of G is either of the form

$$w_{n_1}^1 \times w_{n_2}^2 \times w_{n_3}^3 \times \cdots, g^1 \times C_2 \times C_3 \times \cdots, \quad \text{or}$$

$$w_{n_1}^1 \times w_{n_2}^2 \times \cdots \times w_{n_i}^i \times g^{i+1} \times C_{i+2} \times \cdots$$

where $w_n^i \in W(\epsilon_{n_{i-1}}^{i-1}, C_i)$, $g^i \in G'(\epsilon_{n_{i-1}}^{i-1}, C_i)$, $\epsilon_{n_i}^i = \epsilon_{n_{i-1}}^{i-1}/n_i q$, and $w_{n_i}^i$ is in the slice of C_i at a distance of p/q from the base of C_i.

Suppose M' is a continuum in the Hilbert cube which is the sum of a subcollection G' of G. First we show that if M' intersects each of $S' \times C_2 \times C_3 \times \cdots$ and $S'' \times C_2 \times C_3 \times \cdots$ where S' and S'' are different slices of C_1 parallel to its base, then G' is not upper semicontinuous. There is a sequence of slices $S_1, S_2, \cdots$ each at an irrational distance from the base of C_1 converging to a slice S_0 at a rational distance from this base and such that each S_i separates S' from S''. Then G' is not upper semicontinuous at any of its elements in $S_0 \times C_2 \times C_3 \times \cdots$.

Suppose M' lies in $S \times C_2 \times C_3 \times \cdots$ but intersects two elements of $G(1, C_1)$ in the slice S. Then there is a sequence $g_1, g_2, \cdots$ of elements of $G'(1, C_1)$ in this slice converging to an element w of $W(1, C_1)$ in this slice and such that M' intersects and contains each $g_i \times C_2 \times C_3 \times \cdots$. But G' is not upper semicontinuous at any element in $w \times C_2 \times C_3 \times \cdots$.

We have found that if G' is continuous, the first coordinates of any two elements of G' are the same. Here $w_{n_i}^i$, g^i, and C_i are regarded as coordinates of an element of G'. A similar argument shows that the other coordinates are also equal. Hence, G' has only one element.

QUESTION. For some continua M (for an arc but not for a Hilbert cube) the following statement is true: For each nondegenerate monotone lower semicontinuous decomposition G of M there is a subcontinuum M' of M which is the sum of an infinite subcollection G' of G such that G' is continuous. It would be interesting to know for what types of continua this statement is true. The statement with "lower" replaced by "upper" is not true for any nondegenerate compact continuum.

If we had not been interested in getting a connected M, we would not have needed to suppose that G is monotone. We would have been able to conclude that the collection of elements of G in M is continuous. Consider the following result.

THEOREM 2. *Suppose G has uncountably many elements and is a lower semicontinuous decomposition of a locally compact separable metric space X. There is an uncountable subcollection G' of G such that G' is continuous and the sum of the elements of G' is compact.*

PROOF. Since X has a countable covering $\{D_i\}$ by open sets with compact closures and each element of G is covered by a finite subcollection of $\{D_i\}$, there is a finite subcollection of $\{D_i\}$ that covers uncountably many elements of G. Hence, with no loss of generality

we suppose that X is compact.

The methods of Theorem 1 reveal that there is a compact set M_0 which is the sum of uncountably many elements of G such that if g_1, g_2 are two elements of G in M_0, then $H(g_1, g_2) < 1$.

Again these methods reveal that M_0 contains two mutually exclusive compact subsets C_1^1, C_2^1 such that each is the sum of uncountably many elements of G and such that if g_1, g_2 are elements of G in C_i $(i = 1, 2)$, then $H(g_1, g_2) < 1/2$. Let $M_1 = C_1^1 + C_2^1$.

In general, M_i is the sum of 2^i mutually exclusive compact sets C_1^i, C_2^i, $\cdots$, C_{2i}^i such that C_j^i is the sum of uncountably many elements of G and if g_1, g_2 are two elements of G in C_j^i, $H(g_1, g_2) < 1/2^i$. Each C_j^i contains two of the compact sets C_r^{i+1}, C_s^{i+1} in M_{i+1}.

The collection G' of elements of G in $M' = M_0 \cdot M_1 \cdots$ is continuous. Also, G' has uncountably many elements since it is a Cantor set with respect to its elements as points.

Although this concludes the proof of Theorem 2, it is of interest that if X is compact, and $N(\epsilon)$ is the sum of the elements g of G at which G fails to be upper semicontinuous by ϵ (for each i there is a g_i in G such that $\rho(g, g_i) < 1/i$, $g_i \not\subset V(g, \epsilon)$), then $N(\epsilon)$ is closed and contains no open set. Hence the sum of the elements of G at which G is continuous is a dense G_δ set.

In a bicompact connected T_1 space it does not follow that for each pair of points there is a continuum irreducible between them. Hence, Theorem 3 shows that not each bicompact T_1 space is a space $g(X)$ of the type referred to in statement 6 of the preceding section *Transformations and decompositions*. The following result follows from the same type of argument given in the proof of Theorem 1.

THEOREM 3. *If G is a monotone lower semicontinuous decomposition of a connected compact metric space X and A, B are two nonnull closed subsets of X, there is a continuum M in X which is irreducible with respect to being a continuum which is the sum of elements of G and intersects each of A, B.*

If M_A and M_B denote the sum of the elements of G in M which intersect A and B respectively, then each of $M - M_A$, $M - M_B$, and $M - (M_A + M_B)$ is connected.

4. **Decompositions whose elements are continuous curves.** In this section we shall be interested in a lower semicontinuous decomposition G of a closed planar set such that the elements of G are nondegenerate continuous curves. (A continuous curve is a compact locally connected continuum.) We find that G must be upper semicontinuous also. Example 1 shows that the hypothesis of Theorem 4

must include the condition that the sum of the elements of G is planar.

THEOREM 4. *Suppose G is a lower semicontinuous decomposition of a closed planar set such that each element of G is a nondegenerate continuous curve. Then G is also upper semicontinuous.*

PROOF. Suppose E_1, E_2, E_3 are three point sets. We say that the arc α is *hooked* with respect to E_1, E_2, E_3 if it contains two arcs α_1, α_2 with at most an end point in common such that α_1 intersects all three of E_1, E_2, E_3 and α_2 intersects two of them. Compare this notion of being hooked with the idea of a sequence of chains being hooked as given by Anderson on page 650 of [1].

On the assumption that G is not upper semicontinuous we show that there is a decreasing sequence of compact continua M_1, M_2, $\cdots$ and three decreasing sequences of domains D_{i1}, D_{i2}, $\cdots$ ($i = 1, 2, 3$) such that:

(a) M_j is the sum of uncountably many elements of G.

(b) $M_1 \cdot M_2 \cdot \cdots$ is an element α_0 of G which is an arc.

(c) Each element of G in M_j which is an arc is hooked with respect to D_{1j}, D_{2j}, D_{3j}.

(d) $\overline{D}_{ij+1} \subset D_{ij}$ and $D_{i1} \cdot D_{i2} \cdot \cdots$ is a point p_i with $p_1 \neq p_2 \neq p_3 \neq p_1$.

But since the arc α_0 cannot be hooked with respect to the three points p_1, p_2, p_3 there is an integer j such that α_0 is not hooked with respect to D_{1j}, D_{2j}, D_{3j}. Hence the assumption that G is not upper semicontinuous will have led us to a contradiction.

Suppose g is an element of G at which G is not upper semicontinuous. Then there is a sequence of elements g_1, g_2, $\cdots$ converging to a closed set W which properly contains g. From Theorem 1 there is a compact continuum M in W which properly contains g, is the sum of elements of G, and is such that if g', g'' are two elements of G in M, then the Hausdorff distance $H(g', g'') < 1$.

Let G_M be the collection of elements of G in M. We now establish two properties of G_M that we shall use.

PROPERTY 1. All but a countable number of elements of G_M are arcs. Since the plane does not contain uncountably many triods, all but a countable number of elements of G_M are arcs or simple closed curves. If G_M contained uncountably many simple closed curves, it would contain three such that some one of the three separated the other two from each other in the plane. This is impossible for the sum of the three simple closed curves would belong to the inferior limit of a sequence of plane continua which does not intersect the sum.

PROPERTY 2. G_M is not upper semicontinuous. If G_M were upper

semicontinuous, it would not contain infinitely many elements of the sequence $g_1, g_2, \cdots$. Anderson has shown [1] that if G' is a continuous collection of nondegenerate continuous curves filling a compact continuum M', then M' contains the interior of a circle. But M cannot contain the interior of a circle since it is a subset of the inferior limit of the sequence $g_1, g_2, \cdots$ which has infinitely many elements which do not intersect M. Hence, G_M is not upper semicontinuous.

We note that if M' is any subcontinuum of M which is the sum of uncountably many elements of G, then the collection of elements of G in M' has properties 1 and 2.

Let α be an arc in M which is an element of G, and D_{11}, D_{21}, D_{31} be three mutually exclusive domains intersecting α. We show that there is a continuum M_1 in M which is the sum of uncountably many elements of G and such that each arc in M_1 which is an element of G is hooked with respect to D_{11}, D_{21}, D_{31}.

Let $D_1, D_2, \cdots, D_n$ be a chain of open topological disks covering α such that $D_i \cdot D_j$ is connected or null according as i and j are or are not adjacent and each of D_{11}, D_{21}, D_{31} contains a link of the chain which intersects α. We suppose that D_{n_i} intersects α, $D_{n_i} \subset D_{i1}$, and $n_1 < n_2 < n_3$. For convenience we suppose that $D_{n_1}, D_{n_2}, D_{n_3}$ are the interiors of circles with centers on the x-axis and that if L is the part of the x-axis to the left of D_{n_1} and R is the part to the right of D_{n_3}, then $L + R$ does not intersect any link of the chain.

By Theorem 1 there is a continuum M' in M such that M' is covered by $D_1, D_2, \cdots, D_n$, M' contains α, and M' is the sum of an uncountable subcollection G' of G such that each element of G' will intersect each of D_{n_1}, D_{n_3}. Each element will also intersect D_{n_2} because D_{n_2} is between D_{n_1} and D_{n_3}.

In each element g of G', select an arc h_g which is irreducible from $\overline{D}_{n_1}$ to $\overline{D}_{n_3}$ and denote the collection of all such arcs by H. The elements of H have a natural linear order—for example, h' is above h'' if it lies in the complementary domain of $L + D_{n_1} + h'' + D_{n_2} + R$ which is unbounded from above.

Since G' is not upper semicontinuous, there is a converging sequence $g_2', g_3', \cdots$ of elements of G' and two elements g_0', g_1' of G' which belong to the inferior limit of $g_2', g_3', \cdots$. Let h_i be the element of H in g_i'. Since some subsequence of $h_2, h_3, \cdots$ is monotone, we suppose that h_{i+1} is above h_i if $i > 1$.

In the linear order of H, one of the sets h_0, h_1 is separated from all but a finite number of elements of $h_2, h_3, \cdots$ by some element of $h_0, h_1, h_2, \cdots$. For convenience we suppose that h_j separates h_0 from each element of $h_r, h_{r+1}, \cdots$.

Let N be a connected neighborhood in D_{n_2} which contains a point of h_0 but no point of h_j. Since G' is lower semicontinuous, some element g_k' of g_r', g_{r+1}', $\cdots$ intersects N. Then the component of $g_k' - (D_{n_1} + D_{n_3})$ that contains h_k also intersects $\overline{D}_{n_1}$, D_{n_2}, and $\overline{D}_{n_3}$ but not N, while another component of it intersects N. Now if M_1 is a continuum sufficiently close to g_k' (and there is one sufficiently close by Theorem 1) and the sum of uncountably many elements of G', then each element of G' in M_1 which is an arc is hooked with respect to D_{n_1}, D_{n_2}, and D_{n_3} and hence with respect to D_{11}, D_{21}, D_{31}.

Now that we have obtained D_{11}, D_{21}, D_{31}, and M_1, all we need is an iteration of the preceding argument to get the three sequences D_{i1}, D_{i2}, $\cdots$ ($i = 1, 2, 3$) and the sequence M_1, M_2, $\cdots$. Once M_n, D_{1n}, D_{2n}, D_{3n} are found, we consider an arc α in M_n which is an element of G. Let D_{jn+1} ($j = 1, 2, 3$) be an open set of diameter less than $1/n$ which intersects α and whose closure lies in D_{jn}. We use a $1/n$-chain covering α and show the existence of M_{n+1} by the same methods used to show the existence of M_1. The reason that condition (b) is satisfied is that an arc is the only nondegenerate snake-like continuous curve. (A continuum is snake-like [4] if for each positive number ϵ it can be covered by an ϵ-chain.)

Theorem 4 does not hold if we omit some of the conditions in the hypothesis. We cite some examples to illustrate this.

EXAMPLE 3. If the requirement were not made that the elements of G be nondegenerate, we could have let G be the collection of points on the y-axis from $(0, 0)$ to $(0, 1)$ and the collection of vertical intervals of length 1 with lower ends on the sect $(a, b]$ where $a = (0, 0)$ and $b = (1, 0)$. Then G is lower semicontinuous but not upper semicontinuous.

EXAMPLE 4. It is essential to suppose that the elements of G be continuous curves and not just continua. Let M be a nondegenerate hereditarily indecomposable continuum. Such continua have been described by Knaster [6], Moise [7], and Bing [2; 3]. Kelley has shown [5] that there is a continuous decomposition G' of such a continuum into nondegenerate continua. Let g be an element of G' and G'' be a continuous decomposition of g into nondegenerate continua. Then the collection consisting of elements of G'' and elements of G' other than g is lower semicontinuous but not upper semicontinuous.

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UNE PROPRIÉTÉ TOPOLOGIQUE DES DOMAINES DE RUNGE

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Nous dirons qu'un domaine X de l'espace numérique complexe C^n est un *domaine de Runge* si:

1. X est un domaine d'holomorphie.

2. Toute fonction holomorphe sur X est limite uniforme sur tout compact de polynômes.

On sait que, si $n = 1$, X est simplement connexe. Nous allons généraliser ce résultat:

THÉORÈME. *Le $n^{ème}$ nombre de Betti d'un domaine de Runge de C^n est nul.*

DÉMONSTRATION. Soit $C^n(X)$ l'espace vectoriel des formes différentielles $\omega = f(z_1, \cdots, z_n) \, dz_1 \wedge \cdots \wedge dz_n$, où f est holomorphe sur X. Une telle forme est toujours fermée, i.e. $d\omega = 0$. Soit $B^n(X)$ le sous-espace de $C^n(X)$ formé des éléments ω qui sont de la forme $d\alpha$, où α est une forme différentielle holomorphe de degré $n-1$. D'après le Théorème 1 de [2], qui s'applique à cause de l'hypothèse 1, l'espace quotient $C^n(X)/B^n(X)$ est isomorphe à $H^n(X, \mathbf{C})$, et tout revient donc à montrer que $B^n(X) = C^n(X)$.

Munissons $C^n(X)$ de la topologie de la convergence compacte. Alors:

(a) $B^n(X)$ est *fermé* dans $C^n(X)$.