SYMMETRIC POLYNOMIALS WITH NON-NEGATIVE COEFFICIENTS

L. KUIPERS AND B. MEULENBEDL

1. Introduction. Brunn [1] proved a theorem on a determinant (an alternant) the elements of which are elementary symmetric functions of positive variables. This theorem reflected by us in a sharper form and applied to polynomials (Theorem 1) is the basis of further investigation. A special case is Theorem 2, important applications of which are Theorems 3 and 4. In §2 we prove these results and in §3 we give some examples. In §4 the foregoing is applied to absolutely monotonic functions; the result is Theorem 5, a generalization of a theorem of Rosenbloom [2]. In §5 an extension of Theorem 3 is deduced (Theorem 6) by considering a function of two variables.

2. Let $S_j$ be the elementary symmetric function of $n$ variables $x_1, x_2, \ldots, x_n$, defined by

$$S_j = \sum x_1 x_2 \cdots x_j$$

for $j = 1, 2, \ldots, n$; 

$$S_0 = 1; S_j = 0$$

for $j = -1, -2, \ldots$ and $j > n$.

**Theorem 1.** The determinant $(S_{k_{ij}})$, $i = 1, 2, \ldots, q; j = 1, 2, \ldots, q$, with

$$k_{i,m} - k_{i,m+1} = k(m) > 0,$$

$$k_{m+1,i} - k_{m,i} = k^*(m) > 0,$$

for $i = 1, 2, \ldots, q; m = 1, 2, \ldots, q - 1,$


$\text{can be written as a symmetric polynomial in } x_1, x_2, \ldots, x_n \text{ with non-negative coefficients.}$

**Proof.** Obviously the determinant in question is a symmetric polynomial in the considered variables.

For $n = 1$ the determinant equals zero or unity or a power of $x_1$. Now applying induction we assume the assertion to be true for $n - 1$ and prove the truth for the case $n$. Therefore we put

$$S_k = S'_k + x_n S'_{k-1},$$

where $S'_k$ differs from $S_k$ in referring to the variables with $x_n$ left out. Substituting (2) in the determinant we can expand this in increasing powers of $x_n$, hence

Received by the editors May 12, 1952 and, in revised form, April 4, 1954.

88
(3) \[ (S_{kij}) = \sum_{k=0}^{q} A_k x_n^k, \]

where \( A_k \) is a sum of determinants the elements of which are elementary symmetric functions of \( x_1, x_2, \ldots, x_{n-1} \). The element indices satisfy (1), hence from our assumption it follows that each \( A_k \) is a polynomial in \( x_1, x_2, \ldots, x_{n-1} \) with non-negative coefficients. Because of (3) our assertion is proved.

An immediate consequence of Theorem 1 is

**Theorem 2.** Let \( \sigma_j = (-1)^j S_j \). Then the determinant

\[ (\sigma_{m+i-k_j}) \ (i = 0, 1, \ldots, q; j = 0, 1, \ldots, q; 0 = k_0 < k_1 < \cdots < k_d) \]

multiplied by the factor \( (-1)^M \), where

\[
M = m + (m - k_1) + (m - k_2) + \cdots + (m - k_0) + 1 + 2 + \cdots + q,
\]

is expressible as a symmetric polynomial in \( x_1, x_2, \ldots, x_n \) with non-negative coefficients.

Now we prove the following

**Theorem 3.** Let

\[ f_{h+1}(x) = a_{h0} + a_{h1}x + \cdots + a_{h,n+p}x^{n+p} \quad (h = 0, 1, \ldots, n - 1), \]

where \( n \geq 2 \) and \( p \geq 0 \), be \( n \) polynomials with real coefficients such that all determinants \( D \) of the \( n \)th order, taken from the matrix \( |a_{ij}|, i = 0, 1, \ldots, n - 1; j = 0, 1, \ldots, n + p \), are non-negative.

If for \( n \) variables \( x_1, x_2, \ldots, x_n \), with \( x_i \neq x_j \) for \( i \neq j \), we put \( V = V(x) = \text{the determinant} (x_i), i = 1, 2, \ldots, n; j = 0, 1, \ldots, n - 1 \), then the expression

\[ (f_i(x_j))/V. \]

can be written as a symmetric polynomial in \( x_1, x_2, \ldots, x_n \) with non-negative coefficients.

**Proof.** From a theorem of Garbieri [3] it follows that (4) is equal to the determinant of \( (n+p+1) \)th order

\[
(B_{ij}, B_{ij} = a_{ij} \quad (i = 0, 1, \ldots, n - 1; j = 0, 1, \ldots, n + p),
\]

\[
B_{ij} = \sigma_{i-j} \quad (i = n, n + 1, \ldots, n + p; j = 0, 1, \ldots, n + p),
\]

where \( \sigma_j \) is defined in Theorem 2. By expanding (5) in terms of the \( (p+1) \)-line minors of the last \( p+1 \) rows (let \( i \) denote the rows) we see that \( (B_{ij}) \) is the sum of a number of expressions each of which is
a product of a determinant $D$, the corresponding determinant $(\sigma_{n+i-k_j})$, $i=1, 2, \ldots, p+1; j=1, 2, \ldots, p+1$; $1 \leq k_1 < k_2 < \cdots < k_{p+1} \leq n+p+1$, and the factor $(-1)^{\mathcal{S}}$ where $\mathcal{S}=(n+1)+(n+2)+\cdots+(n+p+1)+k_1+k_2+\cdots+k_{p+1}$. Now the exponent $M$ (see Theorem 2) related to this last determinant is equal to $(n+1-k_1)+\cdots+(n+1-k_{p+1})+1+2+\cdots+p$, so that $M \equiv \mathcal{S} \pmod{2}$.

From this conclusion and Theorem 2 the assertion follows.

A special case of Theorem 3 (put $f_h(x)=x^{k_h}$, $h=1, 2, \ldots, n$) is

**Theorem 4** (P. C. Rosenbloom [2, p. 459]). If $k_1, k_2, \ldots, k_n$ are integers with $0 \leq k_1 < k_2 < \cdots < k_n$, then

\[(x_i^j)/V; \quad i, j = 1, 2, \ldots, n,\]

is a symmetric polynomial in $x_1, x_2, \ldots, x_n$ with non-negative coefficients.

**Remark.** The last result can also be obtained by application of a theorem of H. Naegelsbach [4].

Acting in this way we find for the expression (6) the determinant

\[
\begin{vmatrix}
S_n & S_{n-1} & S_{n-k_1+1} & S_{n-k_1-1} & \cdots & S_{n-k_n+1} \\
0 & S_n & S_{n-k_2} & S_{n-k_2+1} & \cdots & S_{n-k_n+2} \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & S_1
\end{vmatrix}
\]

whose elements satisfy the conditions of Theorem 1.

**3. Examples.** 1. If for $h=1, 2, \ldots, n; p \geq 0$,

\[f_h(x) = \sum_{i=0}^{n+p} a_i^h x^i \quad \text{with } 0 < a_1 < \cdots < a_n,
\]

then $(f_i(x))/V; i, j=1, 2, \ldots, n$, is a symmetric polynomial in $x_1, x_2, \ldots, x_n$ with non-negative coefficients.

This follows from the fact that the determinants of the $n$th order taken from the matrix

\[
\begin{vmatrix}
a_i^{j-i-2}
\end{vmatrix}; \quad i = 1, 2, \ldots, n; j = 1, 2, \ldots, n+p+1,
\]

divided by the positive number $V(a)$ are polynomials in the positive $a_1, a_2, \ldots, a_n$ with non-negative coefficients, as follows from Theorem 4.

2. If the determinants of the $n$th order taken from the matrix $|a_{ij}|$, $i=1, 2, \ldots, n; j=1, 2, \ldots, n+p$, are positive, and if
where the $k_h$ are integers with $0 \leq k_1 < \cdots < k_n$, then $(A_{ij})/V^2$ $(i, j = 1, 2, \cdots, n)$ is a symmetric polynomial with non-negative coefficients.

Proof. Putting $f_i(x) = \sum_{q=0}^{n+p} a_{iq}x^q$ $(i = 1, 2, \cdots, n)$, we have

$$A_{ij} = \sum_{h=1}^n x_i^h \sum_{q=0}^{n+p} a_{iq}x_h^q = \sum_{h=1}^n x_i^h f_i(x_h),$$

so that

$$(A_{ij}) = (x_i^{k_j})(f_i(x_j)) \quad (i, j = 1, 2, \cdots, n).$$

Application of Theorem 3 completes the proof.

4. Theorem 5. If $f(x)$ and $g(x)$ are power series with non-negative coefficients converging in the interval $0 \leq x < a$, then the expression

$$(7) \quad (-1)^{n-1} \det \left| x_1^1 \cdots x_i^n \cdot f(u x_1) g(v x_i) \cdots x_{i-1} x_{i+1} \cdots x_n \right|/V,$$

$i = 1, 2, \cdots, n,$

can be written as a power series in the variables $x_1, x_2, \cdots, x_n, u, v$ with non-negative coefficients converging in the range

$$0 \leq x_1, x_2, \cdots, x_n < a; \quad 0 \leq u \leq 1; \quad 0 \leq v \leq \frac{1}{a^{n-1}}.$$

Proof. Putting

$$f(x) = \sum_{q=0}^\infty a_q x^q, \quad g(x) = \sum_{m=0}^\infty b_m x^m, \quad a_q \geq 0, \quad b_m \geq 0,$$

then (7) can be expressed as

$$(-1)^{n-1} \sum_{q=0}^\infty a_q u^q \left| x_1^1 \cdots x_i^n \cdot x_i^q g(v x_i) \cdots x_{i-1} x_{i+1} \cdots x_n \right|/V$$

$$= (-1)^{n-1} \sum_{q=0}^\infty \sum_{m=0}^\infty a_q b_m u^q v^m \left| x_1^1 \cdots x_i^n \cdot x_i^q (x_1^1 \cdots x_{i-1} x_{i+1} \cdots x_n)^m \right|/V$$

$$= (-1)^{n-1} \sum_{q=0}^\infty \sum_{m=0}^\infty a_q b_m u^q v^m \left| x_i^m x_i^1 \cdots x_i^m x_i^1 \right|/V$$

$$= \sum_{q-n+2}^\infty \sum_{m=0}^\infty a_q b_m u^q v^m \left| x_i^m x_i^1 \cdots x_i^m x_i^1 \right|/V,$$
so that from Theorem 4 the correctness of this theorem follows.

Remark. For \( n = 3 \) this theorem is a result of Rosenbloom about absolutely monotonic functions \([2]\). (The range of \( u \) and \( v \) given there is not quite correct.)

5. Theorem 6. Let \( F(x, y) \) be the function \( \sum_{i=0}^{n+p} \sum_{j=0}^{n+p} a_{ij} x^i y^j \) \( (p \geq 0) \), with the property that all determinants \( D \) of the \( n \)th order taken from the matrix \( |a_{ij}| \) are non-negative. Let \((x_1, x_2, \cdots, x_n)\) and \((y_1, y_2, \cdots, y_n)\) with \( x_i \neq x_j, y_i \neq y_j \) \( (i \neq j) \) be two sets of variables.

Then the expression

\[
(F(x_i, y_j))/V(x)V(y).
\]

is a polynomial, symmetric in \( x_1, x_2, \cdots, x_n \) as well as in \( y_1, y_2, \cdots, y_n \) with non-negative coefficients.

Proof. On account of another theorem of Garbieri \([3]\) the expression (8) is identical with

\[
(-1)^{p-1}(t_{ij}), \quad (i, j = 0, 1, \cdots, n + 2p + 1),
\]

where

\[
t_{ij} = a_{ij} \quad (i, j = 0, 1, \cdots, n + p)
\]

\[
= \sigma'_{i-j-p-1} \quad (i = 0, 1, \cdots, n + p, j = n + p + 1, \cdots, n + 2p + 1)
\]

\[
= \sigma_{i-j-p-1} \quad (i = n + p + 1, \cdots, n + 2p + 1, j = 0, 1, \cdots, n + 2p + 1),
\]

where \((-1)^{i}\sigma_{i}\) and \((-1)^{j}\sigma_{j}\) are the elementary symmetric functions of \( x_1, x_2, \cdots, x_n \) and \( y_1, y_2, \cdots, y_n \) respectively. We develop the determinant in (9) in terms of the \((p+1)\)-line minors of the last \( p+1 \) rows. The term corresponding with the minor indicated by the column-indices \( k_1, k_2, \cdots, k_{p+1}, \) say \( M \), possesses the sign

\[
(-1)^{n+p+2}+(n+p+3)+\cdots+(n+2p+2)+k_1+k_2+\cdots+k_{p+1} = (-1)^M.
\]

If in \( M \) we replace each \( \sigma_j \) by \((-1)^{i}\sigma_j\), then the new minor \( M' \) has the sign

\[
(-1)^{n-k_1+1}+(n-k_2+1)+\cdots+(n-k_{p+1}+1)+1+2+\cdots+p = (-1)^N.
\]

The complementary minor of \( M \) with elements \( a_{ij} \) and \( \sigma'_k \), say \( \overline{M} \), can be expanded in terms of the \((p+1)\)-line minors of the last \( p+1 \) columns. The term in the expansion of \( \overline{M} \) corresponding with the minor \( N \) indicated by the row-indices \( q_1, q_2, \cdots, q_{p+1} \) is provided with the sign

\[
(-1)^{(n+1)+(n+2)+\cdots+(n+p+1)+q_1+q_2+\cdots+q_{p+1}} = (-1)^P.
\]
In $\mathfrak{H}$ again we replace the $\sigma'$ by $S'$, and the new minor $\mathfrak{H}'$ has the sign

$$(-1)^{(n-q_1+1)+(n-q_2+1)+\cdots+(n-q_p+1)+1+2+\cdots+p} = (-1)^Q.$$  

Thus the determinant in (9) is the sum of terms each of which is a product of 3 determinants $\mathfrak{M}'$, $\mathfrak{N}'$, $D$ and the factor $(-1)^{M+N+P+Q}$. By simple calculation we see that

$$M + N + P + Q \equiv (p + 1)^2 \text{ (mod 2).}$$

As from our assumption the determinants $D$ are non-negative, it follows that each term in the development of (9) has the positive sign, on account of $(-1)^{(p+1)+(p+1)} = 1$.

The application of Theorem 1 to each $\mathfrak{M}'$ and $\mathfrak{N}'$ completes the proof.

References


University of Indonesia and

Technische Hogeschool, Delft