ment \(a\) which suffice. Let \(F = B(t)\), \(B\) any field of characteristic two and \(t\) transcendental over \(B\). If \(f(t)\) denotes an arbitrary element of \(B(t)\), then define \(\alpha\) by \(f(t)\alpha = f(1/t)\), and let \(a = t + 1/t\).

**Bibliography**


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**ON THE CHARACTERISTIC FUNCTION OF A MATRIX PRODUCT**

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In a recent note [1], Roth has proved this result.

**Theorem 1.** Let \(A\) and \(B\) be \(n \times n\) matrices, with elements in a field \(F\), and let

\[
\begin{align*}
\left| xI - A \right| &= a_0(x^2) - x a_1(x^2), \\
\left| xI - B \right| &= b_0(x^2) - x b_1(x^2),
\end{align*}
\]

where \(a_0, a_1, b_0,\) and \(b_1\) are elements in the polynomial ring \(F[x]\). If the rank of \(A - B\) is not greater than unity, then

\[
\left| xI - AB \right| = (-)^n \left[ a_0(x) b_0(x) - x a_1(x) b_1(x) \right].
\]

In his proof, which is essentially a verification, Roth derives some interesting but unnecessary information. Here I present a proof which is shorter, direct, and leads naturally to a more general result involving three matrices.

The essential step in my proof is the observation that if \(A\) is a nonsingular matrix and \(M\) is a matrix of rank 1, then

\[
\left| A + M \right| = \left| A \right| + \sum \Delta_i
\]

where \(\sum \Delta_i\) is a sum of \(n\) determinants, each consisting of \(n-1\) columns of \(A\) and one column of \(M\). This follows from the fact that, \(M\) being of rank 1, any two columns of \(M\) are linearly dependent.

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For the case at hand we have
\[ |xI - A| |xI + B| = |(xI - A)(xI + B)| = |x^2I - AB - x(A - B)| \]
and this determinant is equal to \( |x^2I - AB| \) if \( A - B \) has zero rank, while if \( A - B \) has rank 1, we have
\[ |x^2I - AB - x(A - B)| = |x^2I - AB| = x \sum \Delta_i, \]
where each determinant \( \Delta_i \) has \( n-1 \) columns chosen from \( x^2I - AB \) and one column from \( A - B \). It is observed that the terms of \( x \sum \Delta_i \) contain only odd powers of \( x \). Thus, in either case, \( |x^2I - AB| \) is equal to the even part of \( |xI - A| |xI + B| \). Now
\[ |xI - A| |xI + B| = (-)^n [a_0(x^2) - ax_1(x^2)] [b_0(x^2) + xb_1(x^2)], \]
and the even part is \( (-)^n [a_0(x^2)b_0(y) - xa_1(x^2)b_1(y)] \). Hence, writing \( y = x^2 \), we have
\[ |yI - AB| = (-)^n [a_0(y)b_0(y) - ya_1(y)b_1(y)], \]
and this is Roth's result.

Before extending this result we prove the

**Lemma.** If \( H \) and \( K \) are nonzero square matrices, such that \( xH - K \) is of rank 1, for \( x \) indeterminate over the field \( F \), then either

(i) \( H = uh', \ K = vk' \),

or

(ii) \( H = uh', \ K = vh' \),

where \( u, v, h, k \) are column vectors. Conversely, if \( H \) and \( K \) satisfy (i) and (ii) then \( xH - K \) is of rank 1.

**Proof.** Since \( xH - K \) is of rank 1 for all \( x \), it follows that \( H \) and \( K \) are each of rank 1 and hence are of the form
\[ H = uh', \ K = vk', \]
where \( u, v, h, k \) are column vectors. If we now equate to zero all the two-rowed minors of \( xH - K \), it is easily found that either \( u = v \) or \( h = k \), and this proves the lemma. The converse is obviously true.

From this lemma we proceed to

**Theorem 2.** Let \( A_1, A_2, \text{ and } A_3 \) be \( n \times n \) matrices, such that
\[ |xI - A_i| = a_{0i}(x^2) + xa_{1i}(x^2) + x^2a_{2i}(x^2) \quad (i = 1, 2, 3) \]
and write \( H = A_1 + A_2 + A_3, \ K = A_1A_2 + A_1A_3 + A_2A_3 \). If \( H \) and \( K \) satisfy the lemma, or if \( H = K = 0 \), then
$$\begin{vmatrix} xI - A_1A_2A_3 \end{vmatrix} = a_{01}a_{02}a_{03} + x[a_{11}(a_{02}a_{23} + a_{03}a_{22}) \\
+ a_{12}(a_{01}a_{23} + a_{03}a_{21}) + a_{13}(a_{01}a_{22} + a_{02}a_{21})] \\
+ x^2a_{21}a_{22}a_{23},$$

where $a_{ij} = a_{ij}(x)$.

**Proof.** We have

$$(xI - A_1)(xI - A_2)(xI - A_3) = x^3I - A_1A_2A_3 - x(xH - K).$$

If $H = K = 0$ we have

$$E = \begin{vmatrix} xI - A_1 \end{vmatrix} \begin{vmatrix} xI - A_2 \end{vmatrix} \begin{vmatrix} xI - A_3 \end{vmatrix} = \begin{vmatrix} x^3I - A_1A_2A_3 \end{vmatrix}.$$  

If $xH - K$ is of rank 1 for all $x$, we have

$$E = \begin{vmatrix} x^3I - A_1A_2A_3 \end{vmatrix} - x \sum \Delta_i,$$

where each determinant $\Delta_i$, since it consists of $n - 1$ columns of $x^3I - A_1A_2A_3$ and 1 column of $xH - K$, expands into a polynomial each term of which involves $x$ to the power $3k$ or $3k + 1$ for some integer $k$. Now $x \sum \Delta_i$ is a polynomial, each term of which involves $x$ to a power $3k + 1$ or $3k + 2$. Thus, in either case, $\begin{vmatrix} x^3I - A_1A_2A_3 \end{vmatrix}$ is equal to the sum of the terms of $\begin{vmatrix} xI - A_1 \end{vmatrix} \begin{vmatrix} xI - A_2 \end{vmatrix} \begin{vmatrix} xI - A_3 \end{vmatrix}$ which involve powers of $x^3$. If we pick out these terms and replace $x^3$ by $x$ the result follows.

**Reference**


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