

SEPARATION PROPERTIES OF CONVEX CONES¹

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1. **Introduction.** If A and B are convex subsets of a topological linear space E , we say that A and B can be *separated by a hyperplane* provided E admits a continuous linear functional f , not identically zero, such that $\sup_{A} f \leq \inf_{B} f$. Except in rather special cases, the known separation theorems require that A and B have no common point, and thus fail to cover the interesting case in which A and B are closed convex cones whose intersection and common vertex is the origin ϕ . This case is discussed in the present note. Our main results, extending theorems of Aronszajn [1] and Tagamlitzki [6], are proved in §2; §3 contains examples showing that these results cannot be substantially improved.

In addition to standard notation and terminology, we employ the following: A ϕ -cone in a topological linear space is a closed convex cone having vertex ϕ ; for a ϕ -cone A , A' will denote the linear subspace $A \cap -A$. Set-theoretic sum and difference are indicated by \cup and \setminus respectively, $+$ and $-$ being reserved for the linear operations. For points x and y of a linear space, $[x, y] \equiv \{tx + (1-t)y : 0 \leq t \leq 1\}$, $]x, y[\equiv \{tx + (1-t)y : 0 < t < 1\}$, etc.

2. **The separation theorems.** Our main results, (2.5) and (2.7), will be based on several propositions about ϕ -cones. The first of these can be proved by modification of an argument in [5, p. 452] (see also [4, p. 78]):

(2.1) *If A and B are ϕ -cones in a topological linear space, A is locally compact, and $A \cap B = \{\phi\}$, then $A - B$ is closed.*

(2.2) *If A is a ϕ -cone in a locally convex topological linear space E and Q is a nonempty compact convex subset of A which misses A' , then E admits a continuous linear functional f such that $f \geq 0$ on A and $f > 0$ on Q .*

PROOF. Let \mathcal{U} be the family of all convex open sets $U \ni \phi$ such that $Q + U$ misses A' , and for each $U \in \mathcal{U}$ let A_U be the convex hull of $(A \setminus A') \cup (Q + U)$. An elementary argument (similar to that of [5, p. 459]) shows that for some $V \in \mathcal{U}$, $\phi \notin A_V$. Since the interior of A_V is nonempty, it then follows from the basic separation theorem

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([4, p. 69] and [5, p. 456]) that E admits a continuous linear functional f , not identically zero, such that $f \geq 0$ on A_V . It is easy to verify that $f \geq 0$ on A and $f > 0$ on Q .

(2.3) *If A is a locally compact ϕ -cone in a locally convex topological linear space E , then there are locally compact ϕ -cones A_1 and A_2 in E such that $A'_1 = A_1$, $A'_1 = \{\phi\} = A_1 \cap A_2$, and $A_1 + A_2 = A$.*

PROOF. A' is a locally compact linear subspace of E , hence is finite-dimensional, and thus admits a complementary closed linear subspace L in E . With $A_1 = A'$ and $A_2 = A \cap L$, it can be verified that the sets A_i have the stated properties.

The proof of the next proposition is due to J. Dieudonné and the author, independently. Dieudonné also observed that it could be used to prove the theorem (2.5).

(2.4) *If A_2 is a locally compact ϕ -cone in a locally convex topological linear space E , and $A'_2 = \{\phi\}$; then there is a compact convex set $J \subset E \setminus \{\phi\}$ such that $A_2 = [0, \infty [J$.*

PROOF. By local convexity of E and local compactness of A_2 , there is a convex subset K of E such that ϕ is interior to K and $A_2 \cap K$ is compact. Let F be the intersection of A_2 with the boundary of K and let J be the closed convex hull of F . Then clearly $A_2 = [0, \infty [J$, and compactness of J follows from a result of Bourbaki [4, p. 80]. Now if $\phi \in J$, then ϕ is an extreme point of J , for $J \subset A$ and $A' = \{\phi\}$. Then from a theorem of Milman [4, p. 84] it follows that $\phi \in F$, a contradiction completing the proof.

We remark that (2.3) and (2.4) have valid converses which do not require local convexity of E .

A very special case of the theorem below was given by Tagamlitzki [6].

(2.5) THEOREM. *Suppose A and B are ϕ -cones in a locally convex topological linear space E , A is locally compact, and $A \cap B = \{\phi\}$. Then E admits a continuous linear functional f such that $f < 0$ on $A \setminus A'$, $f = 0$ on $A' \cup B'$, and $f \geq 0$ on $B \setminus B'$.*

PROOF. By (2.1), $B \setminus A$ is closed, and hence is a ϕ -cone. Let A_1 and A_2 be as in (2.3) and let J be as in (2.4). Then $(B - A)' = B' - A' = B' - A_1$, so that if $-J$ intersects $(B - A)'$ it must be true that B' intersects $J + A_1$. Since $A \cap B = \{\phi\}$ and $J + A_1 \subset A_2$, this is impossible, and it follows that $-J$ misses $(B - A)'$. Thus, by (2.2), E admits a continuous linear functional f such that $f \geq 0$ on $B - A$ and $f > 0$ on $-J$. It is easy to verify that f has the desired properties.

Now if, in (2.5), B is also locally compact, then E admits a continuous linear functional g such that $g < 0$ on $B \setminus B'$, $g = 0$ on $B' \cup A'$,

and $g \geq 0$ on $A \setminus A'$; $f - g$ is then a linear functional < 0 on $A \setminus A'$, $= 0$ on $A' \cup B'$, and > 0 on $B \setminus B'$. This stronger type of separation is also available in another setting, which we now discuss. The lemma needed is

(2.6) *Suppose E is a complete metric linear space, C is a closed convex subset of E , $\phi \in C$, and x_α is a sequence of points of C . Then there are a point p of C and a sequence t_α of positive numbers such that for each i , $[p - t_i x_i, p + t_i x_i] \subset C$. If F is a set of linear functionals on E such that every member of F which is constant on $\{x_1, x_2, \dots\}$ is constant on C , then $f p \in] \inf_C f, \sup_C f [$ for each $f \in F$.*

PROOF. Let ρ be a complete invariant metric on E . For each i , choose t_i such that $0 < t_i < 2^{-i-1} > \rho(\phi, t_i x_i)$. Then the series $\sum_1^\infty t_i x_i$ converges to a point $p \in C$, and for each i it is true that $[p - t_i x_i, p + t_i x_i] \subset C$. Now consider an arbitrary $f \in F$. If f is constant on $\{x_1, x_2, \dots\}$, then of course $f p \in] \inf_C f, \sup_C f [$. If for some i , $f x_i > 0$, then $\inf_C f \leq f(p - x_i) < f p < f(p + x_i) \leq \sup_C f$. A similar statement holds if $f x_i < 0$ for some i , so the proof is complete.

(2.7) THEOREM. *Suppose E is a separable normed linear space, A and B are ϕ -cones in E , A is locally compact, and $A \cap B = \{\phi\}$. Then E admits a continuous linear functional f such that $f < 0$ on $A \setminus A'$, $f = 0$ on $A' \cup B'$, and $f > 0$ in $B \setminus B'$.*

PROOF. Let $D = B - A$. By (2.1), D is a ϕ -cone. Let C be the set of all $g \in E^*$ such that $g \geq 0$ on D . C is a closed convex subset of E^* , and by [2, p. 124], C is separable in the w^* -topology. Hence by (2.6) there is an $f \in C$ such that $f x \in] \inf_{\sigma \in C} g x, \sup_{\sigma \in C} g x [$ for each $x \in E$. For $x \in D \setminus D'$ we have $\inf_{\sigma \in C} g x \geq 0$ and (by (2.2)) $\sup_{\sigma \in C} g x > 0$, whence $f x > 0$. Thus f is a continuous linear functional on E such that $f \geq 0$ on D and $f > 0$ on $D \setminus D'$. Since $D' = (B - A)' = B' - A'$, it is easy to verify that f has the desired properties.

If L is a linear space and p is a positively homogeneous subadditive functional on L , then, by the Hahn-Banach theorem, L admits a linear functional $f \leq p$. Aronszajn [1] showed that if $p x + p(-x) = n x > 0$ for each $x \in L \setminus \{\phi\}$, and L is separable under the norm n , then L admits a linear functional f such that $f x < p x$ whenever $x \in L \setminus \{\phi\}$. Bonsall [3] showed by an example that the assumption of separability cannot be discarded. Aronszajn's result follows easily from (2.7), by application of (2.7) to the space $L \times R$. (For the type of argument needed, see [5, p. 462].)

3. Remarks and examples. Let E be either the hyper-Hilbert space \mathcal{L}^{∞}_1 or the complete separable locally convex metric linear space (s) of

all sequences of real numbers [2, p. 10], and let B be the set of all non-negative members of E . Then B is a ϕ -cone and $B' = \{\phi\}$, but E does not admit any linear functional f such that $f > 0$ on $B \setminus \{\phi\}$. This shows that E must be taken normed and separable in (2.7), and helps to clarify the role played by local compactness in (2.5).

The separation theorems (2.5) and (2.7) assert more than the possibility of separating A and B by a hyperplane, for in each case we have not merely $f \leq 0$ on B but also $f < 0$ on $B \setminus B'$. We shall now develop an example which shows that without local compactness of at least one of the ϕ -cones involved, even the weakest sort of separation may not be possible.

A subset of a linear space is said to be *linearly bounded* provided its intersection with every line is contained in some bounded subset of the line. For later use, we mention the following fact, whose proof is left to the reader:

(3.1) *If E is a topological linear space, M is a hyperplane in $E \setminus \{\phi\}$, and J is a convex subset of M , then $[0, \infty [J$ is closed if and only if J is closed and linearly bounded.*

Now if E is an arbitrary nonreflexive separable Banach space, then E contains a disjoint pair of bounded closed convex sets which cannot be separated by any hyperplane [5, p. 881]; application of (3.1) produces in E a pair of ϕ -cones A and B which cannot be separated by any hyperplane, even though $A \cap B = \{\phi\}$. In reflexive spaces, on the other hand, disjoint bounded closed convex sets can always be separated by a hyperplane. Nevertheless, we have

(3.2) *Hilbert space H contains a disjoint pair of linearly bounded closed convex sets which cannot be separated by any hyperplane.*

PROOF. We may regard H as $H_1 \times H_2$, where each H_i is a sequential Hilbert space. Let Q be a compact convex subset of H_2 which contains the origin ϕ_2 , but cannot be supported by a hyperplane at ϕ_2 . Let p be a point of H_2 such that $]0, \infty [p$ misses $-Q$. Let the points δ_i of H_1 be given by $\delta_i^i = 1$ and $\delta_i^j = 0$ for $i \neq j$, and let a_α and b_α be sequences of positive numbers, with $b_\alpha \rightarrow 0$. (A further restriction on a_α will be added later.) Let $C = \{(x, \phi_1) : x \in H_1 \text{ and } x^i \in [0, a_i] \text{ for each } i\}$ and let D be the closed convex hull of $\{(a_n \delta_n, q + b_n p) : n \text{ a positive integer, } q \in Q\}$. It can be verified that C and D are linearly bounded closed convex subsets of $H = H_1 \times H_2$.

Now suppose f is a continuous linear functional on H such that $\sup_C f \leq \inf_D f$. Then for each n , $f(a_n \delta_n, \phi_2) \leq f(a_n \delta_n, q + b_n p)$ for all $q \in Q$, whence $f(\phi_1, q) \geq 0$ for all $q \in Q$. Since ϕ_2 is a nonsupport point of Q in H_2 , it follows that $f(\phi_1, y) = 0$ for all $y \in H_2$. But then $f(a_i \delta_i, \phi_2) \leq f(a_j \delta_j, \phi_2)$ for all i and j , and it follows that f must be identically

zero. We still must show that, with appropriate specification of a_α , C misses D .

Consider an arbitrary point (x, ϕ_2) of C . If this is also a point of D , there must be positive numbers t_j^i ($1 \leq i, j < \infty$) and points q_j of Q such that $u_\alpha \rightarrow x$ and $v_\alpha \rightarrow \phi_2$, where for each j it is true that $\sum_{i=1}^{\infty} t_j^i = 1$, $u_j = \sum_{i=1}^{\infty} t_j^i a_i \delta_i$, and $v_j = (\sum_{i=1}^{\infty} t_j^i b_i) p + q_j$. Since $v_\alpha \rightarrow \phi_2$, Q is compact, and $]0, \infty [p + Q$ misses ϕ_2 , it is necessary that $\lim_{j \rightarrow \infty} \sum_{i=1}^{\infty} t_j^i b_i = 0$. Since each b_i is positive, that implies $\lim_{j \rightarrow \infty} t_j^i = 0$ for each i ; and with $u_\alpha \rightarrow x$, this implies $x = \phi_1$. Thus with no further restriction on the sequences a_α and b_α , it must be true that $C \cap D \subset \{(\phi_1, \phi_2)\}$. Now the square of the distance from u_j to ϕ_1 is $\sum_{i=1}^{\infty} (t_j^i a_i)^2$, and for this to be less than 1 it must be true that $t_j^i < a_i^{-1}$ for all i . Since $\sum_{i=1}^{\infty} t_j^i = 1$ for each j , then if a_α is any sequence of positive numbers such that $\sum_{i=1}^{\infty} a_i^{-1} < 1$, the resultant sets C and D have no points in common. This completes the proof.

From (3.1) and (3.2) we now have

(3.3) *Hilbert space contains a pair of ϕ -cones A and B which cannot be separated by a hyperplane, even though $A' = B' = A \cap B = \{\phi\}$.*

A slightly more careful argument would have shown that in (3.3), A and B can be so constructed that $x \cdot y > 0$ whenever

$$\{x, y\} \subset A \cup B \setminus \{\phi\}.$$

In closing, a word about the role played in separation theorems by the assumption that the sets are disjoint. The basic separation theorem [4, p. 69; 5, p. 456] asserts that if C and D are convex and the interior of C is nonempty and misses D , then C and D can be separated by a hyperplane. Here $C \cap D$ need not be empty. But compactness will not suffice for theorems of this type, as is seen from easily constructed examples of compact convex sets C and D in Hilbert space which cannot be separated by a hyperplane, even though, in the first case, D is merely an extreme point of C , and, in the second case, D is a line segment such that $C \cap D$ is a point of C at which C can be supported by a hyperplane. Relevance of these examples to the case of ϕ -cones A and B , and the condition that $A \cap B = \{\phi\}$, can be seen in terms of (3.1). Only in finite-dimensional spaces is the situation especially simple, as described in

(3.4) *Suppose E is a finite-dimensional topological linear space and A_1 and A_2 are convex subsets of E whose union is not contained in any hyperplane. For each i , let A_i^0 be the interior of A_i , relative to the smallest linear manifold containing it. Then A_1 and A_2 can be separated by a hyperplane if and only if A_1^0 misses A_2^0 .*

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A NOTE ON HAAR MEASURE¹

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We prove a direct sum theorem for Haar measure which is a generalization, to arbitrary locally compact topological groups, of a theorem for Lie groups due to Mostow [3]. It is well known that the Haar integral on a *compact* group may be obtained by a simple averaging process. On a *vector* group, the Haar integral is essentially the Lebesgue integral. On a *discrete* group it is summation. Our theorem asserts that the formation of a Haar integral on any locally compact group must be a composite of these three processes. Hence no locally compact group exists for which the Haar integral must be constructed in an essentially more novel manner.

THEOREM. *Given any locally compact topological group G , there exists a compact subgroup K , a subspace E which is homeomorphic to an n -dimensional Euclidean space, and a discrete subset D , such that the mapping $\theta(a, b, c) = abc$ is a homeomorphism of the Cartesian product $D \times E \times K$ with G , and carries the product measure $m_D \times m_E \times m_K$ over to m_G . Here m_D is discrete measure in D , m_E is Lebesgue measure in E relative to suitable coordinates, m_K is Haar measure in K , and m_G is left Haar measure in G . (Note that E need not be a subgroup.)*

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