BIRKHOFF'S PROBLEM 111

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It is well known that the doubly stochastic matrices of order n are precisely the convex combinations of permutation matrices of order n. Problem 111 of Garrett Birkhoff's *Lattice theory* asks for an infinite-dimensional version of this fact. First we show that an infinite doubly stochastic matrix has a positive diagonal. The referee has simplified the proof by use of a result credited to N. G. de Bruijn by Everett and Whaples, *Representations of sequences of sets*, Amer. J. Math. vol. 71 (1949) p. 287. Then we observe, what is rather obvious, that the doubly stochastic matrices are not a closed set in L_∞ norm. The appropriate norm for (a_{ij}) is max $(\sup_i \sum_j |a_{ij}|, \sup_j \sum_i |a_{ij}|)$. In this norm the elements of the convex closure of the permutation matrices are determined by the doubly stochastic conditions

(a) all entries are non-negative;

(b) the sum of each row and each column is unity, together with the requirement

(c) for $\epsilon > 0$ there is $n = n(\epsilon)$ such that in each row or column the sum of the n largest entries is at least $1-\epsilon$. If the matrix be thought of as a matrix of transition probabilities, (c) says that single transitions are uniformly finite with probability 1.

We restate de Bruijn's theorem (in a formally weaker version; but the generalization offers no difficulty). A *diagonal* of a square matrix is a set of entries including just one from each row and just one from each column. A *line* is a row or a column. A matrix is *line-finite* if each line has only finitely many nonzero entries.

THEOREM 1 (de Bruijn). *A line-finite matrix has a nonzero diagonal if and only if each n rows (columns) have, collectively, nonzero entries in at least n columns (rows).*

THEOREM 2. *For any doubly stochastic matrix (a_{ij}) there is a matrix (b_{ij}) satisfying the conditions of de Bruijn's theorem, such that $b_{ij} \leqq a_{ij}$ for each i, j, $b_{ij} \geqq 0$.*

PROOF. Delete from the ith row all nonzero entries but some finite set summing to more than $1-3^{-i}$, and do the same to the columns. Each n rows (columns) originally summed to n, and still sum to more

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than $n-1$; hence their nonzero entries cannot all be contained in $n-1$ or fewer columns (rows).

We observe that any matrix (a_{ij}) such that all entries are nonnegative and each line sums to at most 1 is an L_∞ limit of finite convex combinations of permutation matrices (matrices whose nonzero entries are just a diagonal of ones). By finite induction, a doubly stochastic matrix with only finitely many different entries is such a finite combination. Then the approximation is obvious.

Let X be the Banach space of absolutely line-summable matrices with bounded line sums, normed by the upper limit of line sums. In X we have

THEOREM 3. *The convex closure of the set of permutation matrices is the set of all doubly stochastic matrices which satisfy* (c).

PROOF. If $A=(a_{ij})$ is the limit in norm of a sequence $\{A^n\}$ of finite convex combinations of permutation matrices, then for each $\epsilon>0$ there is an m such that $\|A^m-A\|<\epsilon$; and each row or column in A^m has at most p nonzero entries. The p corresponding entries in the corresponding line in A must then sum to at least $1-\epsilon$.

Conversely, if A satisfies (c), then for each $4\epsilon>0$ there is a matrix $B=(b_{ij})$ with at most $n(\epsilon)$ nonzero entries in each line, $b_{ij}=a_{ij}$, $\|A-B\|<\epsilon$. Choose a natural number $m>n(\epsilon)/\epsilon$. Let c_{ij} be the largest fraction $p/m<b_{ij}$, or 0 if $b_{ij}=0$. Thus each line sum in (c_{ij}) is at least $1-2\epsilon$ and at most $1-1/m$. Then arrange all the lines in a simple sequence $\{L_k\}$. Inductively, the sum of each row (column) is some q/m; and there are infinitely many columns (rows) with indices greater than k, whose sums are less than 1. Hence we can conclude by adding entries $1/m$ in L_k to produce a doubly stochastic matrix D with only finitely many different entries, such that $\|D-A\|<4\epsilon$.

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