Then, for $G \simeq K$ it is necessary and sufficient that R be cyclically equivalent to $A^n(a_1, a_2)$ where $A(a_1, a_2)$ is a primitive element in the free group $F(a_1, a_2)$.

References

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A THEOREM ON COMMUTATIVE POWER ASSOCIATIVE LOOP ALGEBRAS[1]

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Let L be a loop, written multiplicatively, and F an arbitrary field. Define multiplication in the vector space A, of all formal sums of a finite number of elements in L with coefficients in F, by the use of both distributive laws and the definition of multiplication in L. The resulting *loop algebra* $A(L)$ over F is a linear nonassociative algebra (associative, if and only if L is a group).

An algebra A is said to be power associative if the subalgebra $F[x]$ generated by an element x is an associative algebra for every x of A.

THEOREM. *Let $A(L)$ be a loop algebra over a field of characteristic not 2. A necessary and sufficient condition that $A(L)$ be a commutative, power associative algebra is that L be a commutative group.*

PROOF. Assume that $A(L)$ is a commutative, power associative algebra. Clearly L must be commutative and $x^2 \cdot x^2 = (x^2 \cdot x) \cdot x$ for all x of $A(L)$. Under the hypothesis that the characteristic of F is not 2, a linearization[2] of this power identity yields

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[2] See A. A. Albert, *On the power associativity of rings*, Summa Brasiliensis Mathematicae vol. 2, no. 2, pp. 21–32.

$$\sum_{6}(xy + yx)(zw + wz) = \sum_{4}\left[\sum_{3}(zw + wz)y\right]x, \tag{1}$$

for all x, y, z, w of $A(L)$. The sums in (1) are taken over all possible selections of the symbols involved and $\sum_k$ is the sum of k terms. We now restrict our attention to elements of L in $A(L)$ and set $w=z$ in (1). Thus we obtain,

$$\begin{aligned} 4z^2(xy) + 8(xz\cdot zy) = {} & (z^2y)x + (z^2x)y + 2(yz\cdot z)x + 2(xz\cdot z)y \\ & + 2(zy\cdot x)z + 2(zx\cdot y)z + 2(xy\cdot z)z, \end{aligned} \tag{2}$$

as a necessary condition for the power associativity of $A(L)$.

The left member of (2) consists of at most two distinct elements of L. Hence the resolution of the right member into two distinct elements of L requires that two of the three elements $(zy\cdot x)z$, $(zx\cdot y)z$, $(zy\cdot z)z$ be equal. This, together with the commutativity of L, implies that $(xy)z=x(yz)$ for all x, y, z of L.

The proof of the sufficiency is obvious.

The situation is expectedly different if the characteristic of F is two. For example, let L be a commutative loop such that $x^2=1$ (1 the identity element of L) for all x of L. Then if $\alpha=\sum_{i=1}^{n} a_ix_i$, where $a_i\in F$ and $x_1=1$,

$$\alpha^2 = \left[\sum_{i=1}^{n} a_i^2\right] x_1;$$

hence, the power associativity of the loop algebra $A(L)$ is a trivial verification.

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