THE SPLITTING OF CERTAIN SOLVABLE GROUPS

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Let $G$ be a finite group. We shall designate the commutator subgroup of $G$ by $G^2 = [G, G]$; this is the group generated by all commutators $[g, h] = ghg^{-1}h^{-1}$. Inductively $G^n = [G^{n-1}, G]$ is defined to be the group generated by commutators of elements of $G$ with elements of $G^{n-1}$; and $G^*$ will designate $\cap_{n=1}^\infty G^n$. It should be recalled that $G$ is nilpotent if $G^* = E$, the subgroup consisting of the identity element, or equivalently, if $G$ is the direct product of $p$-groups.

Our object here is to show that when $G^*$ is Abelian then there is a nilpotent group $X$ so that $G = XG^*$ where $X \cap G^* = E$. If there are two such splittings of $G$ into $XG^*$ and $YG^*$ then $Y$ and $X$ are conjugates by an element of $G^*$. If $x$ is in the center of $X$ then $x$ does not commute with any of its conjugates. As a consequence of the properties of the splitting it will follow that if $G$ has no center and $G^*$ is Abelian, then both $G$ and its group of automorphisms are contained in the holomorph of $G^*$.

We shall also give an example to show that the hypothesis that $G^*$ be nilpotent instead of Abelian is insufficient to insure a splitting of $G$ in this fashion.

**The splitting of $G$.** In order to show the existence of the splitting mentioned above we first prove the following fact.

**Lemma.** If $G/G^*$ is cyclic, that is, if $G$ is generated by $G^*$ and an element $x$, and if $G^*$ is Abelian, then every element of $G^*$ is of the form $[x, k]$ for some $k \in G^*$. Thus the map sending $k$ into $[x, k]$ is a 1-1 map of $G^*$ onto itself.

**Proof.** To prove this we shall use the following easily verified rules for commutators (cf. [2, p. 60]):

$[a, bc] = [a, b][a, c]^b$ where $g^b$ denotes $bgb^{-1}$,

$[ab, c] = [b, c]^a[a, c],$

and

$[a, b] = [b, a]^{-1}.$

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Then remembering that $G^*$ is an Abelian normal subgroup and that $G^2 = G^*$ we have for $g, h$ in $G^*$

$$[x, gh] = [x, g][x, h]^g = [x, g][x, h]$$

and therefore also

$$[x, g^{-1}] = [x, g]^{-1} = [g, x]$$

since $e = [x, g^{-1}g] = [x, g^{-1}][x, g]$.

Now the elements of $G$ are of the form $gx^r, hx^s$ for $r$ and $s$ integers and hence $[gx^r, hx^s] = [gx^r, h][gx^r, x^s] = [x^r, h][g, x^s]$. But $[x^r, h] = [x^{r-1}, h][x, h] = [x^{r-1}, h^s][x, h]$ and therefore by an induction argument $[x^r, h] = [x, h]$ for some $h$ in $G^*$. Also $[g, x^s] = [x^r, g]^{-1}$ and therefore $[g, x^s] = [x, g]$ for $g$ in $G^*$. It follows that every commutator and hence in view of (1) every element of $G^2$ is of the form $[x, k]$ for some $k$ in $G^*$ as the lemma asserts. That the map sending $k$ into $[x, k]$ is a 1-1 map of $G^*$ onto itself follows readily from this.

**Corollary.** If $H$ is a normal subgroup of $G$ contained in $G^*$ then $[x, H] = H$ where $[x, H]$ denotes the set of commutators $[x, h]$ for $h \in H$. If $K$ is the group generated by $x$ and $H$ then $K$ is not nilpotent and in fact $K^* = H$.

We can now prove the splitting theorem.

**Theorem 1.** If $G$ is a finite group so that $G^*$ is Abelian then $G$ contains a proper subgroup $X$ such that $G^* \cap X = E$, $G = G^*X$, and consequently $X$ is isomorphic to $G/G^*$ and is nilpotent.

**Proof.** $G^*$ is normal in $G$. We shall first consider the case where $G^*$ is minimal normal in $G$, that is $G^*$ does not properly contain any normal subgroup of $G$ other than $E$. Since $G$ is not nilpotent the $\Phi$ subgroup of $G$ (cf. [2, p. 114]) does not contain $G^2$. Therefore there is a minimal set of generators of $G$, $g_1, \cdots, g_k$, where at least one of the generators, say $g_k$, is in $G^2$. Then $g_1, \cdots, g_{k-1}$ generate a proper subgroup $K$ of $G$. Since $G/G^*$ is nilpotent, $g_1G^*, \cdots, g_{k-1}G^*$ generate $G/G^*$ (cf. [2, p. 114] again) and $G = G^*K$. Then $K \cap G^*$ is normal in $K$ and in $G^*$, hence in $G$. Since $K$ is a proper subgroup, $K \cap G^*$ must be $E$ and the theorem is proved when $G^*$ is minimal normal.

If $G^*$ is not minimal normal then we are going to show the existence of a subgroup $H$ properly contained in $G^*$ such that $[G, H] = H$. This is clearly true if $G^*$ has order not a power of a prime; hence suppose $G^*$ has order a power of a prime $p$. Since $G/C^*$ is a direct product of $p$-groups, $G$ has a normal non-nilpotent (cf. [1, pp. 98–102]) subgroup $Q$ containing $G^*$ so that $Q/G^*$ has order a power of a prime $q \neq p$. Hence there is an element of $q$ power order not in the
centralizer $Z$ of $G^*$. Since $Z$ is normal in $G$ and $G/Z$ is the direct product of $p$-groups, there is a central element of $G/Z$ of order a power of $q$ and consequently a normal subgroup $K$ of $G$ generated by $Z$ and an element $x$ of order a power of $q$. $x$ does not commute with all the elements of $G^*$; therefore $K$ is not nilpotent and we have $E \neq K^* \subseteq G^*$, $K^*$ normal in $G$. The elements of $K$ are of the form $x^r z$ for $r$ integral and $z$ in $Z$. Therefore if $g \in G^*$, $[x^r z, g] = [x^r, g]$ and we see that if $L$ is the group generated by $x$ and $K^*$, then $L^* = K^*$. Now we can apply the corollary to the lemma to see that if $H$ is any normal subgroup contained in $K^*$ then $[x, H] = H$.

Now if $K^* \neq G^*$, then $K^*$ is the desired subgroup such that $[G, K^*] = K^*$. If $K^* = G^*$ then any normal subgroup $H$ of $G$ contained in $G^*$ has the property that $[G, H] = H$ since $[x, H] = H$. In either event we can proceed by induction to finish the proof of the theorem. For let $H \neq E$ be properly contained in $G^*$ such that $[G, H] = H$. Then by an induction argument $G$ has a proper subgroup $K$ so that $G/H = K/H \cdot G^*/H$ or $G = K G^*$ with $K \cap G^* \subseteq H$. Then $[K, H] = H$ since $[G, H] = H$ and $K^* \subseteq G^* \cap K \subseteq H$; hence $K^* = H \neq E$ and by the induction argument $K = X K^*$ where $X \cap K^* = E$. Finally $G = K G^* = X G^*$ and $X \cap G^* \subseteq K \cap G^* \subseteq H$; hence $X \cap G^* \subseteq H \cap X = K^* \cap X = E$ and the theorem is proved.

Remark. We shall give here an example to show that the above type of splitting is in general impossible when $G^*$ is nilpotent even if $G/G^*$ is Abelian. For $p$ a prime not 2 let $H$ be a group of order $p^4$, generated by elements $a$, $b$, and $c$; $a$ and $b$ of order $p$, $c$ of order $p^2$, and $c^p = [a, b]$, $[c, a] = [c, b] = e$, the identity. Let $h$ be an automorphism of $H$ sending $a$ into $a^{-1}$, $b$ into $b^{-1}$, and $c$ into $c$; and let $G$ be the holomorph of $H$ with $h$ of order $2p^4$. Then $G^*$ consists of the group of order $p^3$ generated by $a$ and $b$. Since $c$ is of order $p^2$ and $c^p = [a, b]$ the impossibility of a splitting as in the theorem is clear.

On the conjugacy of the complements of $G^*$. If $G = AB$ where $A$ and $B$ are subgroups whose intersection is the identity we shall call $A$ a complement of $B$ in $G$. Our main result here is then the following.

**Theorem 2.** If $G^*$ is Abelian and if $X$ and $Y$ are two complements of $G^*$, then for some $h \in G^*$, $X = h Y h^{-1}$.

**Proof.** First suppose that $G^*$ is a minimal normal subgroup of $G$; then $G^*$ has order a power of some prime $p$. Let $x$ be of order $q$ prime to $p$ in the center of $X$. If $x$ is not in the centralizer of $G^*$ then $x$ and $G^*$ generate a normal subgroup $R$ of $G$ which is not nilpotent and therefore $R^* = G^*$ by the minimality condition on $G^*$. It follows from
the corollary to the lemma of the last section that every element $h$ of $G^*$ is of the form $xgx^{-1}g^{-1}$ for some $g$ in $G^*$.

Now there is a $y \neq e$ in $R \cap Y$ such that $y = x^{-1}h$ and therefore $y = gx^{-1}g^{-1}$. Suppose $Y \neq gXg^{-1}$; then since $G = G^*(gXg^{-1})$ there is a $k$ in $Y$ so that $k = hgmg^{-1}$ for some $h$ in $G^*$, $h \neq e$, and $m$ in $X$. But then since $y$ is a conjugate of $x^{-1}$, $x$ in the center of $X$, it follows that $[k, y] = [hmg^{-1}, y] = [h, y]$; hence $[[k, y] \cdots y] = [[h, y] \cdots y] \neq e$ since $[y, G^*] = G^*$. But this is a contradiction of the nilpotency of $Y$ and we conclude that $Y = gXg^{-1}$ when $x$ is not in the centralizer of $G^*$.

If $x$ is in the centralizer of $G^*$ then $x$ is in the center of $G$ and since $x$ is in every Sylow $q$ group of $G$, $x$ is in $Y$. Then by an induction argument the theorem is true in $G/(x)$ and from this the theorem follows for $G$ when $G^*$ is minimal normal.

The general case now follows easily from this. As in the proof of Theorem 1 there is an $x$ and a normal subgroup $H$ of $G$ properly contained in $G^*$ such that $[x, H] = H$. Without loss in generality $x$ can be taken in $X$. By an induction argument, $Y/H = (gH)X/H(gH)^{-1}$ for some $gH$ in $G^*/H$ and hence if $g$ is an element of $gH$ then $gXg^{-1}$ is a complement of $K^*$ in the proper subgroup $K$ generated by $Y$ and $H$. But $[K, H] = H = K^*$ and by the induction assumption again there is an $h \in H$ so that $h(gXg^{-1})h^{-1} = Y$. This completes the proof of the theorem.

**Remark.** If $h$ is in $G^*$ then either $h$ is in the center of $G$ or $hXh^{-1} \neq X$. For if $hXh^{-1} = X$ then $X$ is normal in the group generated by $X$ and $h$; so also is $(h)$. Hence $[x, h] \subset X \cap (h) = E$ and $h$ is in the center of $G$.

**Remark.** If $x$ is in the center of $X$ then $x$ does not commute with any of its conjugates. For if $y \neq x$ is a conjugate of $x$, then it is clear that $y = hxh^{-1}$ for some $h \in G^*$. If $x$ and $y$ commute, then $x$ commutes with $[h, x] \in G^*$ where $[h, x] \neq e$ since $y \neq x$. Hence $[x, G^*] \neq G^*$. Let $K$ be the group generated by $x$ and $G^*$; then $K^*$ is properly contained in $G^*$ and, being normal in $G$, $K$ contains a normal subgroup $H$ of $G$ so that either $[H, x] = H$ or $[H, x] = E$. Then by an induction argument assuming the statement true in $G/H$, we see that since $y = hxh^{-1}$ then $h$ must be in $H$ and consequently $[h, x]$ is in $H$. Since $[h, x] \neq e$ and commutes with $x$ it is not possible that $[H, x] = H$. On the other hand $[H, x]$ cannot be $E$ for then $[h, x]$ would be $e$. We thus get a contradiction by assuming that $x$ can commute with one of its conjugates.

On the group of automorphisms of $G$ when $G^*$ is Abelian and $G$ has no center.
Theorem 3. If $G$ is a group with no center and $G^*$ is Abelian, then both $G$ and $A$, the group of automorphisms of $G$, are contained in the holomorph $R$ of $G^*$. Furthermore if $D$ and $F$ are complements of $G^*$ in $A$, then there is an $h \in G^*$ so that $hDh^{-1} = F$.

Proof. That $G$ is in $R$ follows from Theorem 1; for $G = G^*X$ where $G^*$ is normal in $G$ and $G \cap X = E$.

Now $G^*$ is a characteristic subgroup of $G$ and therefore every automorphism of $G$ maps $G^*$ into itself. Suppose $m$ is an automorphism of $G$ which commutes with all the elements of $G^*$. Let $L$ be the holomorph of $G$ and $m$ and let $Z$ be the centralizer of $G^*$ in $L$. Then $Z$ is normal in $L$ and hence for every $h \in G^*$, $[mh, G] \subset Z \cap G = G^*$. In view of Theorem 2 there is an $h_0$ in $G^*$ so that $mh_0$ maps $X$ into itself; that is, $[mh_0, X] \subset X$. Therefore $[mh_0, X] \subset X \cap G^* = E$ and $mh_0$ commutes with every element of $X$ as well as of $G^*$; that is, $mh_0$ is the identity automorphism in $G$ or $m$ is the same as the inner automorphism determined by $h_0^{-1}$. It follows that $A/G^*$ is isomorphic to a subgroup $M$ of automorphisms of $G^*$, $M$ containing $X$ as a normal subgroup. Then $N$, the holomorph of $M$ with $G^*$ contains a copy of $G$ and the centralizer of this copy of $G$ is $E$. Then since $N$ and $A$ are both groups of automorphisms of $G$ having the same order, $A$ is isomorphic to $N$ and hence $A$ is a subgroup of $R$ as the theorem asserts.

We now show that if $D$ and $F$ are two complements of $G^*$ in $A$ then there is an $h \in G^*$ so that $F = hDh^{-1}$. It is clear that $D \cap G$ is normal in $D$ and is a complement of $G^*$ in $G$; hence there is an $h \in G^*$ so that $h(D \cap G)h^{-1} = F \cap G$. But $F \cap G$ determines $F$ completely; for if $F \cap G$ were normal in $F$ and also in $F' \neq F$, then $F \cap G$ would be normal in the group generated by $F$ and $F'$ which must necessarily intersect $G^*$ in a subgroup $Z \neq E$. But then $[Z, F \cap G] \subset (F \cap G)$ and $[Z, F \cap G] \subset G^*$, whence it follows that $[Z, F \cap G] = E$ and $G$ has a nontrivial center contrary to hypothesis. It follows that $F \cap G$ determines $F$ completely, and then $hDh^{-1}$ must be $F$. This completes the proof of the theorem.

Bibliography