APPROXIMATION TO IRRATIONALS BY CLASSES OF RATIONAL NUMBERS

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Hurwitz [**3**] proved that there exist infinitely many rational numbers a/b for every irrational ξ such that

$$(1) \qquad |\xi - a/b| < k/b^2$$

if and only if $k \geqq 1/5^{1/2}$. Scott [**11**] proved that if the fractions a/b are restricted to any one of the three classes (i) a, b both odd, (ii) a even, b odd, or (iii) a odd, b even, the same conclusion holds if $k \geqq 1$. Other proofs of this have been given by Oppenheim [**8**], Robinson [**10**], and Kuipers and Meulenbeld [**6**]. Robinson also showed that if any pair of these classes were used, then $k \geqq 1/2$.

Let $(m, r, s) = 1$; then the set of all fractions a/b in lowest terms for which $a \equiv r$, $b \equiv s \pmod m$ will be denoted by $\langle r, s \rangle$. Descombes and Poitou [**1**; **9**] have investigated the values of k needed for sets $\langle r, s \rangle$. Hartman [**2**] and Koksma [**5**] have considered the problem of a universal constant for all sets of fractions a/b with $a \equiv r$, $b \equiv s \pmod m$ where a, b need not be relatively prime nor is it required that $(r, s, m) = 1$.

We obtain results for other classes of rational numbers.

Let the continued fraction expansion of ξ be $\xi = [d_0, d_1, \cdots]$; then the nth convergent is $a_n/b_n = [d_0, d_1, \cdots, d_n]$ and the nth denominator is d_n. We shall use the following known results [**4**; **8**; **10**].

LEMMA A. *If a_{n-1}/b_{n-1}, a_n/b_n, a_{n+1}/b_{n+1} are three consecutive convergents to ξ, then at least one of them satisfies* (1) *with $k = 1/5^{1/2}$.*

LEMMA B. *Let a_n/b_n, a_{n+1}/b_{n+1} be two consecutive convergents to ξ. Then at least one of them satisfies* (1) *with $k = 1/2$. The same is true with $k = 1$ of one of $(a_{n+1} - a_n)/(b_{n+1} - b_n)$ and $(a_{n+1} + a_n)/(b_{n+1} + b_n)$.*

LEMMA C. *If $(a, b) = 1$ and if $|\xi - a/b| < 1/b^2$, then a/b is either a_n/b_n, $(a_n + a_{n+1})/(b_n + b_{n+1})$, or $(a_n - a_{n-1})/(b_n - b_{n-1})$ for a suitable n.*

LEMMA D. *If a_{n-1}/b_{n-1}, a_n/b_n, a_{n+1}/b_{n+1} are three consecutive convergents, then $a_{n+1} = d_{n+1}a_n + a_{n-1}$, $b_{n+1} = d_{n+1}b_n + b_{n-1}$, where d_n is the nth denominator, and $a_n b_{n+1} - a_{n+1} b_n = \pm 1$.*

If $|\xi - a/b| = c/b^2$, we call c the *approximation coefficient* of a/b (for ξ).

Presented to the Society, May 1, 1954; received by the editors May 20, 1954.

LEMMA E. *The approximation coefficient k_n of the nth convergent a_n/b_n of ξ is given by* $1/k_n = [d_{n+1}, d_{n+2}, \cdots] + 1/[d_n, d_{n-1}, \cdots, d_1]$.

Let C be a certain class of fractions a/b. We shall say that k_0 is the approximation coefficient for the class C if it is true that for every irrational ξ there exist infinitely many rational numbers a/b in C such that (1) holds if and only if $k \geqq k_0$.

LEMMA 1. *Let $m = p^e$ (p an odd prime). Then the approximation coefficient of the class of rational numbers a/b with $(a, m) = 1$ is $1/5^{1/2}$.*

Suppose that $(a_n, p) = p$. Therefore $(a_{n+1}, p) = 1$ by Lemma D. Next $a_{n+2} = a_n + d_{n+2}a_{n+1}$. If $d_{n+2} \geqq 3$, a_{n+1}/b_{n+1} is satisfactory by Lemma E. But if d_{n+2} is 1 or 2, then $(d_{n+2}, p) = 1$ since p divides a_n and not $d_{n+2}a_{n+1}$. If $d_{n+3} \geqq 3$, then as before we see that a_{n+2}/b_{n+2} is satisfactory. Now suppose d_{n+2} and d_{n+3} are both less than 3 and one of them is 2; if $d_{n+2} = 2$, then a_{n+1}/b_{n+1} is satisfactory for by Lemma E, $1/k_{n+1} \leqq [2, 2, 1, \cdots] + [0, \cdots] > 7/3 > 5^{1/2}$; if $d_{n+3} = 2$ then similarly a_{n+2}/b_{n+2} will do. Otherwise a_{n+1}/b_{n+1}, a_{n+2}/b_{n+2}, a_{n+3}/b_{n+3} are three consecutive convergents with numerators prime to p. By Lemma A, at least one is satisfactory.

We have shown that if $(a_n, p) = p$, at least one of the three following convergents is effective. Now either $(a_n, p) = p$, or $(a_{n+1}, p) = p$, or $(a_{n+2}, p) = p$, or $(a_n a_{n+1} a_{n+2}, p) = 1$. In this last case one of a_n/b_n, a_{n+1}/b_{n+1}, a_{n+2}/b_{n+2} will do. Thus we see that in all cases among any six consecutive convergents of ξ at least one is of the type described in the lemma.

That the constant $1/5^{1/2}$ cannot be decreased follows from the known fact that an irrational ξ requires this value if and only if its denominators are ultimately all 1. The set of such numbers is denumerable.

LEMMA 2. *Let $m = 2^e$ ($e \geqq 1$). Then the approximation coefficient of the class of rational numbers a/b with a prime to m is $1/2$.*

This result was proved by Robinson [**10**].

LEMMA 3. *Let $m = p^e q^f$, where p, q are distinct primes and e, f are both positive. Then the approximation coefficient of the class of all rational numbers a/b with $(a, m) = 1$ is 1.*

Let a_n/b_n, a_{n+1}/b_{n+1} be two consecutive convergents of ξ. If either has numerator prime to m, then that fraction approximates sufficiently closely by Lemma E. Otherwise a_n is divisible by either p or q, say p, and a_{n+1} is divisible by q since $(a_n, a_{n+1}) = 1$ by Lemma D.

But then $a_{n+1} \pm a_n$ are prime to m and one of $(a_{n+1} \pm a_n)/(b_{n+1} \pm b_n)$ approximates ξ sufficiently closely by Lemma B.

To prove the converse, let $\xi = [0, p^e, w, z_3 m, z_4 m, \cdots]$ where w is chosen so that $wp^f + 1 \equiv 0 \pmod{q^e}$ and $z_n \to \infty$. Then using Lemma D we see that no convergent has numerator prime to m. Any a/b satisfying (1) with $k=1$ is thus one of $(a_n \pm a_{n+1})/(b_n \pm b_{n+1})$ for some n by Lemma D. But if $\alpha_n = [d_n, d_{n-1}, \cdots, d_1]$, $\beta_n = [d_{n+1}, d_{n+2}, \cdots]$, then $(a_n + a_{n+1})/(b_n + b_{n+1})$ has approximating coefficient $1/\{1 + 1/(\beta_n - 1) + 1/(\alpha_n + 1)\}$ and $(a_{n+1} - a_n)/(b_{n+1} - b_n)$ has approximating coefficient $1/\{1 + 1/(\alpha_{n+1} - 1) - 1/(\beta_{n+1} + 1)\}$. Since $\alpha_n \to \infty$, $\beta_n \to \infty$, the theorem follows. The set of such numbers ξ has the power of the continuum.

LEMMA 4. *For every pair of classes $\langle r, s\rangle$ and $\langle r', s'\rangle$ there is a unimodular linear fractional transformation*

$$z' = (Az + B)/(Cz + D), \tag{2}$$

where A, B, C, D are integers and $|AD - BC| = 1$, such that the class $\langle r, s\rangle$ is sent onto the class $\langle r', s'\rangle$.

It is an elementary result that there exists a transformation which sends a given fraction a/b into a given a'/b'. It is easy to see that any other fraction in $\langle a, b\rangle$ will have its image in $\langle a', b'\rangle$. The inverse transformation shows that the mapping is onto.

LEMMA 5. *The approximation coefficients of the classes $\langle r, s\rangle$ are all the same for a given m.*

This result was stated by Descombes and Poitou [1]. We require the following proof in order to prove the next lemma.

Let k be the approximation coefficient of a class $\langle r, s\rangle$. Then

$$k = \sup_{\xi} \left(\liminf_{a/b} b^2 \left| \xi - a/b \right| \right)$$

where a/b ranges in the class $\langle r, s\rangle$. We shall show that the same value of k is obtained if a/b ranges through the class $\langle r', s'\rangle$.

For if ξ', a'/b' are the images of ξ, a/b and if $b^2|\xi - a/b| = k_0$, then

$$b'^2 \left| \xi' - a'/b' \right| = k_0' = k \left[\frac{Ca'/b' + D}{Ca'/b' + D + Ck/b^2} \right].$$

As $a/b \to \xi$, $q \to \infty$, and $k_0' \to k_0$. Hence

$$\liminf k_0' = \liminf k_0$$

and the lemma follows.

The set of all classes $\langle rt, st\rangle$ for fixed r, s and for all t prime to m will be called the class $\{r, s\}$. If a transformation (2) sends $\langle r, s\rangle$ onto $\langle r', s'\rangle$, it sends $\langle rt, st\rangle$ onto $\langle r't, s't\rangle$. Hence the class $\{r, s\}$ is sent onto $\{r', s'\}$. This implies the next lemma.

LEMMA 6. *The approximation coefficients of the classes $\{r, s\}$ are all the same for a given m.*

THEOREM 7. *Let $(r, s, m)=1$. Then the approximation coefficient of the class of all rational numbers a/b with $a\equiv rt$, $b\equiv st$* (mod m), *where t is an integer depending on a, b, is $m/5^{1/2}$.*

Robinson [10] proved this result for the case $m=2$.

We prove the theorem for the case $s=0$. The proof in general then follows from an argument similar to that used in the proof of Lemma 6. Thus $b=dm$. The rational approximations a/dm to ξ are in one-to-one correspondence with the rational approximations a/d to $m\xi$. Since $|a/d-m\xi|>k/d^2$ is satisfied for infinitely many a/d for each irrational ξ if and only if $k\geqq 1/5^{1/2}$ [2], we see that $|a/dm-\xi|<km/(dm)^2$ holds for infinitely many a/dm if and only if $k\geqq 1/5^{1/2}$.

THEOREM 8. *Let $(r, s, m)=1$. The approximation coefficient of the class of all rational numbers a/b in lowest terms with $a\equiv rt$, $b\equiv st$* (mod m) *where t is an integer depending on a, b, is:*

$m/5^{1/2}$ if $m=p^e$, p an odd prime;

$m/2$ if $m=2^e$, $e\geqq 1$;

m if $m=p^eq^f$, p and q distinct primes, e and f positive.

By Lemma 6 it suffices to prove the theorem when $s=0$. A discussion similar to that in the proof of the preceding theorem shows that this theorem follows from Lemmas 1, 2, and 3.

Our final result is a generalization of Lemmas 1, 2, and 3.

THEOREM 9. *Let $m=uv$ where $(u, v)=1$. Then the set of rational numbers a/b in lowest terms such that $(a, v)=1$ and $(b, u)=1$ has approximation coefficient*

$1/5^{1/2}$ if $m=p^e$, p an odd prime;

$1/2$ if $m=2^e$, $e\geqq 1$;

1 if $m=p^eq^f$, p and q distinct primes, e and f positive.

The proof reverses the argument used to obtain Theorem 8 from Lemmas 1, 2, and 3. Let $r=v$, $s=u$. Then the class $\{r, s\}$ has approximation coefficient km as given in Theorem 8. All fractions a'/b' in $\{r, s\}$ have $a'=av$ and $b'=bu$, where a, b are integral. Hence $|\xi-av/bu|\leqq km/(bu)^2$ and therefore $|(u/v)\xi-a/b|<(u/v)kuv/(bu)^2$ $=k/b^2$.

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