APPROXIMATION TO IRRATIONALS BY CLASSES OF RATIONAL NUMBERS

LEONARD TORNHEIM

Hurwitz [3] proved that there exist infinitely many rational numbers \( a/b \) for every irrational \( \xi \) such that

\[
| \xi - a/b | < k/b^2
\]

if and only if \( k \geq 1/5^{1/2} \). Scott [11] proved that if the fractions \( a/b \) are restricted to any one of the three classes (i) \( a, b \) both odd, (ii) \( a \) even, \( b \) odd, or (iii) \( a \) odd, \( b \) even, the same conclusion holds if \( k \geq 1 \). Other proofs of this have been given by Oppenheim [8], Robinson [10], and Kuipers and Meulenbeld [6]. Robinson also showed that if any pair of these classes were used, then \( k \geq 1/2 \).

Let \((m, r, s) = 1\); then the set of all fractions \( a/b \) in lowest terms for which \( a \equiv r, b \equiv s \pmod{m} \) will be denoted by \( \langle r, s \rangle \). Descombes and Poitou [1; 9] have investigated the values of \( k \) needed for sets \( \langle r, s \rangle \). Hartman [2] and Koksma [5] have considered the problem of a universal constant for all sets of fractions \( a/b \) with \( a \equiv r, b \equiv s \pmod{m} \) where \( a, b \) need not be relatively prime nor is it required that \( (r, s, m) = 1 \).

We obtain results for other classes of rational numbers.

Let the continued fraction expansion of \( \xi \) be \( \xi = [d_0, d_1, \ldots] \); then the \( n \)th convergent is \( a_n/b_n = [d_0, d_1, \ldots, d_n] \) and the \( n \)th denominator is \( d_n \). We shall use the following known results [4; 8; 10].

**Lemma A.** If \( a_{n-1}/b_{n-1}, a_n/b_n, a_{n+1}/b_{n+1} \) are three consecutive convergents to \( \xi \), then at least one of them satisfies (1) with \( k = 1/5^{1/2} \).

**Lemma B.** Let \( a_n/b_n, a_{n+1}/b_{n+1} \) be two consecutive convergents to \( \xi \). Then at least one of them satisfies (1) with \( k = 1/2 \). The same is true with \( k = 1 \) of one of \( (a_{n+1} - a_n)/(b_{n+1} - b_n) \) and \( (a_{n+1} + a_n)/(b_{n+1} + b_n) \).

**Lemma C.** If \( (a, b) = 1 \) and if \( |\xi - a/b| < 1/b^2 \), then \( a/b \) is either \( a_n/b_n, (a_n + a_{n+1})/(b_n + b_{n+1}) \), or \( (a_n - a_{n-1})/(b_n - b_{n-1}) \) for a suitable \( n \).

**Lemma D.** If \( a_{n-1}/b_{n-1}, a_n/b_n, a_{n+1}/b_{n+1} \) are three consecutive convergents, then \( a_{n+1} = d_{n+1}a_n + a_{n-1}, b_{n+1} = d_{n+1}b_n + b_{n-1} \), where \( d_n \) is the \( n \)th denominator, and \( a_nb_{n+1} - a_{n+1}b_n = \pm 1 \).

If \( |\xi - a/b| = c/b^2 \), we call \( c \) the approximation coefficient of \( a/b \) (for \( \xi \)).

Presented to the Society, May 1, 1954; received by the editors May 20, 1954.
Lemma E. The approximation coefficient $k_n$ of the $n$th convergent $a_n/b_n$ of $\xi$ is given by $1/k_n = [d_{n+1}, d_{n+2}, \ldots ] + 1/[d_n, d_{n-1}, \ldots , d_1]$.

Let $C$ be a certain class of fractions $a/b$. We shall say that $k_0$ is the approximation coefficient for the class $C$ if it is true that for every irrational $\xi$ there exist infinitely many rational numbers $a/b$ in $C$ such that (1) holds if and only if $k \geq k_0$.

Lemma 1. Let $m = p^e (p$ an odd prime). Then the approximation coefficient of the class of rational numbers $a/b$ with $(a, m) = 1$ is $1/5^{1/2}$.

Suppose that $(a_n, p) = p$. Therefore $(a_{n+1}, p) = 1$ by Lemma D. Next $a_{n+2} = a_n + d_{n+2}a_{n+1}$. If $d_{n+2} \geq 3$, $a_{n+1}/b_{n+1}$ is satisfactory by Lemma E. But if $d_{n+2}$ is 1 or 2, then $(d_{n+2}, p) = 1$ since $p$ divides $a_n$ and not $d_{n+2}a_{n+1}$. If $d_{n+3} \geq 3$, then as before we see that $a_{n+2}/b_{n+2}$ is satisfactory. Now suppose $d_{n+2}$ and $d_{n+3}$ are both less than 3 and one of them is 2; if $d_{n+2} = 2$, then $a_n/b_n$ is satisfactory for by Lemma E, $1/k_{n+1} \leq [2, 2, 1, \ldots ] + [0, \ldots ] > 7/3 > 5^{1/2}$; if $d_{n+3} = 2$ then similarly $a_{n+2}/b_{n+2}$ will do. Otherwise $a_{n+1}/b_{n+1}$, $a_{n+2}/b_{n+2}$, $a_{n+3}/b_{n+3}$ are three consecutive convergents with numerators prime to $p$. By Lemma A, at least one is satisfactory.

We have shown that if $(a_n, p) = p$, at least one of the three following convergents is effective. Now either $(a_n, p) = p$, or $(a_{n+1}, p) = p$, or $(a_{n+2}, p) = p$, or $(a_{n+1}a_{n+2}, p) = 1$. In this last case one of $a_n/b_n$, $a_{n+1}/b_{n+1}$, $a_{n+2}/b_{n+2}$ will do. Thus we see that in all cases among any six consecutive convergents of $\xi$ at least one is of the type described in the lemma.

That the constant $1/5^{1/2}$ cannot be decreased follows from the known fact that an irrational $\xi$ requires this value if and only if its denominators are ultimately all 1. The set of such numbers is denumerable.

Lemma 2. Let $m = 2^e (e \geq 1)$. Then the approximation coefficient of the class of rational numbers $a/b$ with a prime to $m$ is $1/2$.

This result was proved by Robinson [10].

Lemma 3. Let $m = p^e q^f$, where $p$, $q$ are distinct primes and $e$, $f$ are both positive. Then the approximation coefficient of the class of all rational numbers $a/b$ with $(a, m) = 1$ is 1.

Let $a_n/b_n$, $a_{n+1}/b_{n+1}$ be two consecutive convergents of $\xi$. If either has numerator prime to $m$, then that fraction approximates sufficiently closely by Lemma E. Otherwise $a_n$ is divisible by either $p$ or $q$, say $p$, and $a_{n+1}$ is divisible by $q$ since $(a_n, a_{n+1}) = 1$ by Lemma D.
But then $a_{n+1} \pm a_n$ are prime to $m$ and one of $(a_{n+1} \pm a_n)/(b_{n+1} \pm b_n)$ approximates $\xi$ sufficiently closely by Lemma B.

To prove the converse, let $\xi = [0, p', w, z_{\delta m}, z_{\delta m}, \ldots]$ where $w$ is chosen so that $wp' + 1 \equiv 0 \pmod{q'}$ and $z_n \to \infty$. Then using Lemma D we see that no convergent has numerator prime to $m$. Any $a/b$ satisfying (1) with $k = 1$ is thus one of $(a_{n+1} \pm a_n)/(b_{n+1} \pm b_n)$ for some $n$ by Lemma D. But if $\alpha_n = [d_n, d_{n-1}, \ldots, d_1]$, $\beta_n = [d_{n+1}, d_{n+2}, \ldots]$, then $(a_{n+1} \pm a_n)/(b_{n+1} \pm b_n)$ has approximating coefficient $1/(1 + 1/(\beta_n-1) - 1/(\alpha_n+1))$ and $(a_{n+1} - a_n)/(b_{n+1} - b_n)$ has approximating coefficient $1/(1 + 1/(\alpha_n+1) - 1/(\beta_n+1))$. Since $\alpha_n \to \infty$, $\beta_n \to \infty$, the theorem follows. The set of such numbers $\xi$ has the power of the continuum.

**Lemma 4.** For every pair of classes $\langle r, s \rangle$ and $\langle r', s' \rangle$ there is a unimodular linear fractional transformation

$$z' = (Az + B)/(Cz + D),$$

where $A, B, C, D$ are integers and $|AD - BC| = 1$, such that the class $\langle r, s \rangle$ is sent onto the class $\langle r', s' \rangle$.

It is an elementary result that there exists a transformation which sends a given fraction $a/b$ into a given $a'/b'$. It is easy to see that any other fraction in $\langle a, b \rangle$ will have its image in $\langle a', b' \rangle$. The inverse transformation shows that the mapping is onto.

**Lemma 5.** The approximation coefficients of the classes $\langle r, s \rangle$ are all the same for a given $m$.

This result was stated by Descombes and Poitou [1]. We require the following proof in order to prove the next lemma.

Let $k$ be the approximation coefficient of a class $\langle r, s \rangle$. Then

$$k = \sup_{\xi} \left( \liminf_{a/b} b^2 \left| \xi - a/b \right| \right)$$

where $a/b$ ranges in the class $\langle r, s \rangle$. We shall show that the same value of $k$ is obtained if $a/b$ ranges through the class $\langle r', s' \rangle$.

For if $\xi', a'/b'$ are the images of $\xi$, $a/b$ and if $b^2 \left| \xi - a/b \right| = k_0$, then

$$b'^2 \left| \xi' - a'/b' \right| = k_0' = k \left[ \frac{Ca'/b' + D}{Ca'/b' + D + Ck/b^2} \right].$$

As $a/b \to \xi$, $q \to \infty$, and $k_0' \to k_0$. Hence

$$\liminf k_0' = \liminf k_0$$

and the lemma follows.
The set of all classes \( \langle rt, st \rangle \) for fixed \( r, s \) and for all \( t \) prime to \( m \) will be called the class \{ \( r, s \) \}. If a transformation (2) sends \( \langle r, s \rangle \) onto \( \langle r', s' \rangle \), it sends \( \langle rt, st \rangle \) onto \( \langle r't, s't \rangle \). Hence the class \{ \( r, s \) \} is sent onto \{ \( r', s' \) \}. This implies the next lemma.

**Lemma 6.** The approximation coefficients of the classes \{ \( r, s \) \} are all the same for a given \( m \).

**Theorem 7.** Let \( (r, s, m) = 1 \). Then the approximation coefficient of the class of all rational numbers \( a/b \) with \( a \equiv rt, b \equiv st \) (mod \( m \)), where \( t \) is an integer depending on \( a, b \), is \( m/5^{1/2} \).

Robinson [10] proved this result for the case \( m = 2 \).

We prove the theorem for the case \( s = 0 \). The proof in general then follows from an argument similar to that used in the proof of Lemma 6. Thus \( b = dm \). The rational approximations \( a/dm \) to \( \xi \) are in one-to-one correspondence with the rational approximations \( a/d \) to \( m\xi \). Since \( |a/d - m\xi| > k/d^2 \) is satisfied for infinitely many \( a/d \) for each irrational \( \xi \) if and only if \( k \geq 1/5^{1/2} \) [2], we see that \( |a/dm - \xi| < km/(dm)^2 \) holds for infinitely many \( a/dm \) if and only if \( k \geq 1/5^{1/2} \).

**Theorem 8.** Let \( (r, s, m) = 1 \). The approximation coefficient of the class of all rational numbers \( a/b \) in lowest terms with \( a \equiv rt, b \equiv st \) (mod \( m \)), where \( t \) is an integer depending on \( a, b \), is:

- \( m/5^{1/2} \) if \( m = p^e \), \( p \) an odd prime;
- \( m/2 \) if \( m = 2^e \), \( e \geq 1 \);
- \( m \) if \( m = p^e q^f \), \( p \) and \( q \) distinct primes, \( e \) and \( f \) positive.

By Lemma 6 it suffices to prove the theorem when \( s = 0 \). A discussion similar to that in the proof of the preceding theorem shows that this theorem follows from Lemmas 1, 2, and 3.

Our final result is a generalization of Lemmas 1, 2, and 3.

**Theorem 9.** Let \( m = uv \) where \( (u, v) = 1 \). Then the set of rational numbers \( a/b \) in lowest terms such that \( (a, v) = 1 \) and \( (b, u) = 1 \) has approximation coefficient

- \( 1/5^{1/2} \) if \( m = p^e \), \( p \) an odd prime;
- \( 1/2 \) if \( m = 2^e \), \( e \geq 1 \);
- \( 1 \) if \( m = p^e q^f \), \( p \) and \( q \) distinct primes, \( e \) and \( f \) positive.

The proof reverses the argument used to obtain Theorem 8 from Lemmas 1, 2, and 3. Let \( r = v, s = u \). Then the class \{ \( r, s \) \} has approximation coefficient \( km \) as given in Theorem 8. All fractions \( a'/b' \) in \{ \( r, s \) \} have \( a' = av \) and \( b' = bu \), where \( a, b \) are integral. Hence \( |\xi - av/bu| \leq km/(bu)^2 \) and therefore \( |(u/v)\xi - a/b| < (u/v)kuv/(bu)^2 = k/b^2 \).
Bibliography


University of Michigan