AN INEQUALITY FOR LINEAR OPERATORS BETWEEN L^p SPACES

R. E. FULLERTON

1. **Introduction.** Let R and S be sets of points with completely additive non-negative measure ϕ and ψ defined over σ-rings $\mathcal{J}(R)$, $\mathcal{J}(S)$ of subsets of R and S respectively. Assume also that R and S are σ-finite with respect to ϕ and ψ. For any real number p, $1 \leqq p < \infty$, $L^p(R, \phi)$ will denote the Banach space of real-valued functions $x(\psi)$ for which $\int_R |x(t)|^p d\phi < \infty$ where the norm is defined by $\|x\| = (\int_R |x(t)|^p d\phi)^{1/p}$. $L^q(S, \psi)$ will denote a similar space for the set S with measure ψ. It is known [2; 3] that the space $L^p(R, \phi)$ is equivalent as a Banach space to the space $V^p(R, \phi)$ of completely additive set functions of bounded p-variation where p-var $F(e) = \sup_\pi (\sum_{i=1}^n |F(e_i)|^p/[\phi(e_i)]^{p-1})^{1/p}$ where the supremum is taken with respect to all finite families π of disjoint subsets of R of finite nonzero measure.

If T is a bounded linear operator between two spaces of the above type, $L^p(R, \phi)$ and $L^q(S, \psi)$, it can be shown that T can be represented as the derivative of an integral involving a kernel. Thus $Tx = (d/de)\int_R K(e, t)x(t)d\phi$ where $K(e, t)$ is defined for $e \in F(R)$, $t \in R$, $\int_R K(e, t)x(t)d\phi \in V^q(S, \psi)$ and the symbol d/de applied to a completely additive and absolutely continuous set function denotes the integrable point function associated with it by the Radon-Nikodym theorem. This representation can be obtained by using the standard theorems for representations of linear functionals over spaces of the type $L^p(R, \phi)$ [1].

The problem in connection with this representation which presents a greater difficulty is that of determining necessary and sufficient conditions on the kernel to insure that the operator is one of the desired type and also to find a suitable expression or suitable inequalities for the norm of the operator T in terms of the kernel. Thus if the norm of T lies between two fixed multiples of some expression involving the kernel, this yields a convenient necessary and sufficient condition that T be bounded. In the case in which $1 \leqq p < \infty$, $q = 1$, such bounds were found on the norm of T and in case T was a positive operator, an exact expression for the norm of T was found by the author [2]. In this paper, the same techniques as were used in [2] are applied to the case for $q > 1$ to determine a convenient expression for a lower bound on the norm of T under certain

Presented to the Society, May 1, 1954; received by the editors May 17, 1954.

conditions on p and q and on the measure of S. The result is weaker than the result in the case $q=1$ but it does appear to give a nontrivial lower bound to the norm of the operator T.

2. Inequalities for the norm of the operator.

THEOREM. *Let T be a bounded linear operator from $L^p(R, \phi)$ to $L^q(S, \psi)$, $1 \leqq p, q < \infty$. Then there exists a real-valued function $K(e, t)$ defined on $\mathcal{J}(S) \times R$ such that for each $x \in L^p(R, \phi)$,*

$$Tx = (d/de) \int_R K(e, t) x(t) d\phi$$

where $K(e, t)$ satisfies the following conditions:

(1) *$\int_E K(\cdot, t) d\phi \in V^q(S, \psi)$ for each $E \in \mathcal{J}(R)$, $\phi(E) < \infty$.*

(2) *$K(e, \cdot) \in L^{p'}(R, \phi)$ for each $e \in \mathcal{J}(S)$, $1/p + 1/p' = 1$.*

(3) $\|T\| \leqq q\text{-var}_S \left[\left(\int_R |K(e, t)|^{p'} d\phi \right)^{1/p'} \right]$.

(4) *If $p \leqq q$ and $\psi(S)$ is finite then the integral $\int_R |K(e, t)|^{p'} d\phi$ has bounded q-variation.*

(5) $\left(q\text{-var}_S \int_R |K(e,t)|^{p'} d\phi\right)^{1/p'} / \sup_\pi \left(\sum_{j=1}^m [\psi(e_j)]^{(p'-1)(q-1)} \right)^{1/qp'} \leqq \|T\|$ *where the supremum is taken over all finite disjoint families of sets of S of finite nonzero measure.*

In particular, if $\psi(S) \leqq 1$ and $p \leqq q$ then

$$\left[q\text{-var}_S \int_R |K(e, t)|^{p'} d\phi \right]^{1/p'} \leqq \|T\|.$$

PROOF. If T is bounded linear from $L^p(R, \phi)$ to $L^q(S, \psi)$ and if $Tx = y$, let the operator U be defined from $L^p(R, \phi)$ to $V^q(S, \psi)$ as $Ux = \int_e y(s) d\psi$. U is bounded linear from $L^p(R, \phi)$ to $V^q(S, \psi)$ and has the same norm as T [**2**; **3**] and hence for any fixed $e \in F(S)$, the functional defined by assigning to each $x \in L^p(R, \phi)$ the value of Ux at e yields a linear functional over $L^p(R, \phi)$. Thus, by using the known representation theorems for linear functionals on L^p, [**1**], $Ux(e) = \int_R K(e, t) x(t) d\phi$ where $K(e, \cdot) \in L^{p'}(R, \phi)$ for each $e \in F(S)$. Thus $Tx = (d/de) \int_R K(e, t) x(t) d\phi$ and (2) is proved. If $\gamma_E(t)$ is the characteristic function of $E \in \mathcal{J}(S)$, then $\gamma_E \in L^p(R, \phi)$ and $\int_R K(e, t) \gamma_E(t) d\phi = \int_E K(e, t) d\phi \in V^q(S, \psi)$. Thus (1) is true. The Hölder inequality can be used to prove (3). For $x \in L^p(R, \phi)$,

$$\|Tx\| = \|Ux\| = q\text{-var}_S \int_R K(e, t) x(t) d\phi$$

$$\leqq q\text{-var}_S \left(\int_R | K(e, t) |^{p'} d\phi \right)^{1/p'} \left(\int_R | x(t) |^p d\phi \right)^{1/p}.$$

The principal part of the proof of the rest of the theorem lies in the proof of the inequality (5). Let $\Pi=\{\pi\}$ be the class of all families of disjoint measurable subsets $\pi=\{E_i\}$ of finite nonzero measure in R. It was proved in [2] that the set $B\subset L^p(R, \phi)$ consisting of all functions of the type $z(t)=\sum_{i=1}^n \{[a_i\gamma_{E_i}(t)]/[\psi(E_i)]^{1/p}\}$ is dense in the unit sphere of $L^p(R, \phi)$ where $\gamma_E(t)$ denotes the characteristic function of E, $\sum_{i=1}^n |a_i|^p\leqq 1$, and $\{E_i\}$ ranges over all of Π. Thus $\|T\|=\sup_{z\in B}\|Tz\|$. Let $\{e_j\}$ be any finite family of sets in $\mathcal{J}(S)$ of finite nonzero measure. Then

$$\left\{\sum_{j=1}^m \frac{\left|\int_R K(e_j, t)z(t)d\phi\right|^q}{[\psi(e_j)]^{q-1}}\right\}^{1/q}$$

$$=\left\{\sum_{j=1}^m \frac{\left|\sum_{i=1}^n a_i\left(\int_{E_i} K(e_j, t)d\phi\right)\Big/ [\phi(E_i)]^{1/p}\right|^q}{[\psi(e_j)]^{q-1}}\right\}^{1/q}$$

$$=\left\{\sum_{j=1}^m \left|\sum_{i=1}^n a_i \frac{\int_{E_i} K(e_j, t)d\phi}{[\psi(e_j)]^{1/q'}[\phi(E_i)]^{1/p}}\right|^q\right\}^{1/q} \leqq \|T\|.$$

In particular, for any $e\in F(S)$, $0<\psi(e)<\infty$,

$$\sum_{i=1}^n a_i \frac{\int_{E_i} K(e, t)d\phi}{[\phi(E_i)]^{1/p}} \leqq \|T\|\,[\psi(e)]^{1/q'}$$

for all families $\{E_i\}$, $i=1, 2, \cdots, n$, as described above and for all finite sequences $\{a_i\}$, $i=1, 2, \cdots, n$, of real numbers such that $\sum_{i=1}^n |a_i|^p\leqq 1$. Let $\beta_i=[\int_{E_i}K(e, t)d\phi]/[\phi(E_i)]^{1/p}$, $i=1, 2, \cdots, n$. This sequence $\{\beta_i\}$ can be considered as an infinite sequence by adding zeros after the nth term. Hence, if $\{a_i\}$ is any sequence for which $\sum_{i=1}^\infty |a_i|^p\leqq 1$, $\left|\sum_{i=1}^\infty \beta_i a_i\right|\leqq\|T\|\,[\psi(e)]^{1/q'}$. Hence the sequence $\{\beta_i\}$ represents a linear functional f over l^p of norm not exceeding $\|T\|\,[\psi(e)]^{1/q'}$ and by using the known representation theorems for the norm of such a functional, $\|f\|=\sup_{\sum|a_i|\leqq 1}\left|\sum_{i=1}^\infty \beta_i a_i\right| =(\sum_{i=1}^\infty |\beta_i|^{p'})^{1/p'}\leqq\|T\|\,[\psi(e)]^{1/q'}$. Hence

$$\left(\sum_{i=1}^n \left|\left[\int_{E_i} K(e, t)d\phi\right]\Big/ [\phi(E_i)]^{1/p}\right|^{p'}\right)^{1/p'} \leqq \|T\|\,[\psi(e)]^{1/q'}$$

for all partitions $\{E_i\}$ of R as described above and for any $e\in F(S)$, $0<\psi(e)<\infty$. Hence if the supremum is taken over all such partitions

$$\sup_\pi \left\{ \sum_{i=1}^{n} \left| \frac{\int_{E_i} K(e, t) d\phi}{[\phi(E_i)]^{1/p}} \right|^{p'} \right\}^{1/p'} = \sup_\pi \left\{ \sum_{i=1}^{n} \frac{\left| \int_{E_i} K(e, t) d\phi \right|^{p'}}{[\phi(E_i)]^{p'-1}} \right\}^{1/p'}$$

$$= p'\text{-var}_R \left| \int_E K(e, t) d\phi \right|$$

$$= \left(\int_R | K(e, t) |^{p'} d\phi \right)^{1/p'}$$

$$\leqq \|T\| [\psi(e)]^{(q-1)/q}$$

and hence for any $e \in \mathcal{J}(S)$ of finite nonzero measure,

$$\frac{\left[\int_R | K(e, t) |^{p'} d\phi \right]^q}{[\psi(e)]^{q-1}} \leqq \|T\|^{p'q} [\psi(e)]^{(p'-1)(q-1)}$$

Let $\{e_j\}$ be any finite family of subsets of S of finite nonzero measure, $j = 1, 2, \cdots, m$. Then

$$\sum_{j=1}^{m} \frac{\left[\int_R | K(e_j, t) |^{p'} d\phi \right]^q}{[\psi(e_j)]^{q-1}} \leqq \|T\|^{p'q} \left(\sum_{j=1}^{m} [\psi(e_j)]^{(p'-1)(q-1)} \right),$$

$$\left\{ \sum_{j=1}^{m} \frac{\left[\int_R | K(e_j, t) |^{p'} d\phi \right]^q}{[\psi(e_j)]^{q-1}} \right\}^{1/q} \leqq \|T\|^{p'} \left(\sum_{j=1}^{m} [\psi(e_j)]^{(p'-1)(q-1)} \right)^{1/q}.$$

If the least upper bound of the two quantities is taken with respect to all such families $\{e_j\}$ of S then

$$\sup_\pi \left\{ \sum_{j=1}^{m} \frac{\left[\int_R | K(e_j, t) |^{p'} d\phi \right]^q}{[\psi(e_j)]^{q-1}} \right\}^{1/q}$$

$$= q\text{-var}_\pi \left(\int_R | K(e, t) |^{p'} d\phi \right) \leqq \|T\|^{p'} \sup_\pi \left(\sum_{j=1}^{m} [\psi(e_j)]^{(p'-1)(q-1)} \right)^{1/q}$$

and hence

$$\left(q\text{-var}_\pi \int_R | K(e, t) |^{p'} d\phi \right)^{1/p'} \Big/ \sup_\pi \left(\sum_{j=1}^{m} [\psi(e_j)]^{(p'-1)(q-1)} \right)^{1/qp'} \leqq \|T\|.$$

If S has finite measure and $(p'-1)(q-1) \geqq 1$, the supremum in the denominator is finite and this implies that $\int_R |K(e, t)|^{p'} d\phi$ has bounded q-variation. In particular, if the measure of S is less than one and $(p'-1)(q-1) \geqq 1$, then for each partition $\{e_j\}$, $\sum_{j=1}^{m} [\psi(e_j)]^{(p'-1)(q-1)} \leqq 1$ and

$$\left[q\text{-var}_S\left(\int_R |K(e, t)|^{p'} d\phi\right)\right]^{1/p'} \leqq \|T\|.$$

However, $(p'-1)(q-1) \geqq 1$ implies $q \geqq p'/(p'-1) = p$. Thus if $q \geqq p$ the above inequalities will hold. Hence if $\psi(S) \leqq 1$, $q \geqq p$, the double inequality $[q\text{-var} \int_R |K(e, t)|^{p'} d\phi]^{1/p'} \leqq \|T\| \leqq q\text{-var} (\int_R |K(e, t)|^{p'} d\phi)^{1/p'}$ will hold. Thus if the right member of the inequality is finite the integral represents a bounded linear operator between L^p and V^q, and if the norm of T is finite, the left member of the inequality is finite. It is to be noted that since there is no guarantee that $\int_R |K(e, t)|^{p'} d\phi$ is an additive set function, the theorem of Riesz on the equivalence of the q-variation to the qth integral norm will not apply and the q-variation cannot be replaced by an integral.

In case $p = q$, then $(q-1)(p'-1) = 1$ and in this case

$$\sup_{\pi} \sum_{j=1}^{m} [\psi(e_j)]^{(p'-1)(q-1)} = \psi(S).$$

Hence we have the

COROLLARY. *If S has finite measure and $p = q$, then*

$$\left(\left[q\text{-var}_S \int_R |K(e, t)|^{q'} d\phi\right] \Big/ [\psi(S)]^{1/q}\right)^{1/q'} \leqq \|T\|.$$

REFERENCES

1. S. Banach, *Theorie des opérationes linéaires*, Warsaw, 1934.

2. R. E. Fullerton, *The representation of linear operators from L^p to L*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 689–696.

3. F. Riesz, *Untersuche über Systeme integrierbar Funktionen*, Math. Ann. vol. 69 (1910) pp. 449–497.

UNIVERSITY OF WISCONSIN