AN INEQUALITY FOR LINEAR OPERATORS BETWEEN 
\( L^p \) SPACES

R. E. FULLERTON

1. Introduction. Let \( R \) and \( S \) be sets of points with completely additive non-negative measure \( \phi \) and \( \psi \) defined over \( \sigma \)-rings \( \mathcal{F}(R) \), \( \mathcal{F}(S) \) of subsets of \( R \) and \( S \) respectively. Assume also that \( R \) and \( S \) are \( \sigma \)-finite with respect to \( \phi \) and \( \psi \). For any real number \( p \), \( 1 \leq p < \infty \), \( L^p(R, \phi) \) will denote the Banach space of real-valued functions \( x(\phi) \) for which \( \int_R |x(t)|^p d\phi < \infty \) where the norm is defined by \( \|x\| = \left( \int_R |x(t)|^p d\phi \right)^{1/p} \). \( L^q(S, \psi) \) will denote a similar space for the set \( S \) with measure \( \psi \). It is known \([2; 3]\) that the space \( L^p(R, \phi) \) is equivalent as a Banach space to the space \( V^p(R, \phi) \) of completely additive set functions of bounded \( p \)-variation where \( p \)-var \( F(e) = \sup \left( \sum_{i=1}^n \left| F(e_i) \right|^p / \left[ \phi(e_i) \right]^{p-1} \right)^{1/p} \) where the supremum is taken with respect to all finite families \( \pi \) of disjoint subsets of \( R \) of finite nonzero measure.

If \( T \) is a bounded linear operator between two spaces of the above type, \( L^p(R, \phi) \) and \( L^q(S, \psi) \), it can be shown that \( T \) can be represented as the derivative of an integral involving a kernel. Thus \( Tx = (d/d\phi) \int_R K(e, t)x(t)d\phi \) where \( K(e, t) \) is defined for \( e \in F(R) \), \( t \in R \), \( \int_R K(e, t)x(t)d\phi \subseteq V^q(S, \psi) \) and the symbol \( d/d\phi \) applied to a completely additive and absolutely continuous set function denotes the integrable point function associated with it by the Radon-Nikodym theorem. This representation can be obtained by using the standard theorems for representations of linear functionals over spaces of the type \( L^p(R, \phi) \) \([1]\).

The problem in connection with this representation which presents a greater difficulty is that of determining necessary and sufficient conditions on the kernel to insure that the operator is one of the desired type and also to find a suitable expression or suitable inequalities for the norm of the operator \( T \) in terms of the kernel. Thus if the norm of \( T \) lies between two fixed multiples of some expression involving the kernel, this yields a convenient necessary and sufficient condition that \( T \) be bounded. In the case in which \( 1 \leq p < \infty \), \( q = 1 \), such bounds were found on the norm of \( T \) and in case \( T \) was a positive operator, an exact expression for the norm of \( T \) was found by the author \([2]\). In this paper, the same techniques as were used in \([2]\) are applied to the case for \( q > 1 \) to determine a convenient expression for a lower bound on the norm of \( T \) under certain

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conditions on \( p \) and \( q \) and on the measure of \( S \). The result is weaker than the result in the case \( q = 1 \) but it does appear to give a non-trivial lower bound to the norm of the operator \( T \).

2. Inequalities for the norm of the operator.

**Theorem.** Let \( T \) be a bounded linear operator from \( L^p(\mathbb{R}, \phi) \) to \( L^q(S, \psi) \), \( 1 \leq p, q < \infty \). Then there exists a real-valued function \( K(e, t) \) defined on \( \mathcal{F}(S) \times \mathbb{R} \) such that for each \( x \in L^p(\mathbb{R}, \phi) \),

\[
Tx = \left( \frac{d}{de} \right) \int_R K(e, t)x(t)d\phi
\]

where \( K(e, t) \) satisfies the following conditions:

1. \( \int_E K(\cdot, t)d\phi \in V^q(S, \psi) \) for each \( E \in \mathcal{F}(\mathbb{R}), \phi(E) < \infty \).
2. \( K(e, \cdot) \in L^{p'}(\mathbb{R}, \phi) \) for each \( e \in \mathcal{F}(S), 1/p + 1/p' = 1 \).
3. \( ||T|| \leq q\text{-vars } \left( \int_R |K(e, t)|^{p'}d\phi \right)^{1/p} \).
4. If \( p \leq q \) and \( \psi(S) \) is finite then the integral \( \int_R |K(e, t)|^{p'}d\phi \) has bounded \( q \)-variation.
5. \( (q\text{-vars } \int_R |K(e, t)|^{p'}d\phi)^{1/p'} / \sup_\tau \left( \sum_{i=1}^n [\psi(e_i)]^{(p'-1)(q-1)/q} \right) \leq ||T|| \)

where the supremum is taken over all finite disjoint families of sets of \( S \) of finite nonzero measure.

In particular, if \( \psi(S) \leq 1 \) and \( p \leq q \) then

\[
\left[ q\text{-vars } \int_R |K(e, t)|^{p'}d\phi \right]^{1/p'} \leq ||T||.
\]

**Proof.** If \( T \) is bounded linear from \( L^p(\mathbb{R}, \phi) \) to \( L^q(S, \psi) \) and if \( Tx = y \), let the operator \( U \) be defined from \( L^p(\mathbb{R}, \phi) \) to \( V^q(S, \psi) \) as \( Ux = \int_S \gamma(x)d\phi \). \( U \) is bounded linear from \( L^p(\mathbb{R}, \phi) \) to \( V^q(S, \psi) \) and has the same norm as \( T \) \([2; 3]\) and hence for any fixed \( e \in \mathcal{F}(S) \), the functional defined by assigning to each \( x \in L^p(\mathbb{R}, \phi) \) the value of \( Ux \) at \( e \) yields a linear functional over \( L^p(\mathbb{R}, \phi) \). Thus, by using the known representation theorems for linear functionals on \( L^p \), \([1]\), \( Ux(e) = \int_R K(e, t)x(t)d\phi \) where \( K(e, \cdot) \in L^{p'}(\mathbb{R}, \phi) \) for each \( e \in \mathcal{F}(S) \). Thus \( Tx = \left( \frac{d}{de} \right) \int_R K(e, t)x(t)d\phi \) and (2) is proved. If \( \gamma_E(t) \) is the characteristic function of \( E \in \mathcal{F}(S) \), then \( \gamma_E \in L^{p'}(\mathbb{R}, \phi) \) and \( \int_R K(e, t)\gamma_E(t)d\phi = \int_E K(e, t)d\phi \in V^q(S, \psi) \). Thus (1) is true. The Hölder inequality can be used to prove (3). For \( x \in L^p(\mathbb{R}, \phi) \),

\[
||Tx|| = ||Ux|| = q\text{-vars } \int_R K(e, t)x(t)d\phi \leq q\text{-vars } \left( \int_R |K(e, t)|^{p'}d\phi \right)^{1/p'} \left( \int_R |x(t)|^p d\phi \right)^{1/p}.
\]
The principal part of the proof of the rest of the theorem lies in the proof of the inequality (5). Let $\Pi = \{\pi\}$ be the class of all families of disjoint measurable subsets $\pi = \{E_i\}$ of finite nonzero measure in $R$. It was proved in [2] that the set $B \subseteq L^p(R, \phi)$ consisting of all functions of the type $z(t) = \sum_{i=1}^{n} \left( \frac{a_i \gamma_{E_i}(t)}{\phi(E_i)^{1/p}} \right)$ is dense in the unit sphere of $L^p(R, \phi)$ where $\gamma_{E_i}(t)$ denotes the characteristic function of $E_i$, $\sum_{i=1}^{n} a_i |E_i|^{1/p} \leq 1$, and $\{E_i\}$ ranges over all of $\Pi$. Thus $\|T\| = \sup_{e \in E} \|Tz\|$. Let $\{e_i\}$ be any finite family of sets in $\mathcal{F}(S)$ of finite nonzero measure. Then

$$\left\{ \sum_{j=1}^{m} \left( \int_{R} K(e_j, t) z(t) d\phi \right)^q \right\}^{1/q} = \left\{ \sum_{j=1}^{m} \left( \sum_{i=1}^{n} a_i \left( \int_{E_i} K(e_j, t) d\phi \right) \right) \left( \phi(E_i)^{1/p} \right)^q \right\}^{1/q} \leq \|T\|.$$  

In particular, for any $e \in F(S)$, $0 < \psi(e) < \infty$,

$$\sum_{i=1}^{n} \left( \frac{\int_{E_i} K(e, t) d\phi}{\phi(E_i)^{1/p}} \right)^q \leq \|T\| \left( \|\psi(e)\|^{1/q} \right)^q$$

for all families $\{E_i\}$, $i = 1, 2, \ldots, n$, as described above and for all finite sequences $\{a_i\}$, $i = 1, 2, \ldots, n$, of real numbers such that $\sum_{i=1}^{n} |a_i|^{1/p} \leq 1$. Let $\beta_i = \left( \int_{E_i} K(e, t) d\phi \right) \left( \phi(E_i)^{1/p} \right)^q$, $i = 1, 2, \ldots, n$. This sequence $\{\beta_i\}$ can be considered as an infinite sequence by adding zeros after the $n$th term. Hence, if $\{a_i\}$ is any sequence for which $\sum_{i=1}^{n} |a_i|^{1/p} \leq 1$, $\sum_{i=1}^{n} \beta_i a_i \leq \|T\| \left( \|\psi(e)\|^{1/q} \right)^q$. Hence the sequence $\{\beta_i\}$ represents a linear functional $\varphi$ over $l^p$ of norm not exceeding $\|T\| \left( \|\psi(e)\|^{1/q} \right)^q$ and by using the known representation theorems for the norm of such a functional, $\|\varphi\| = \sup_{e \in F(S)} \left( \sum_{i=1}^{n} \beta_i |a_i| \right)^{1/p'} \leq \|T\| \left( \|\psi(e)\|^{1/q} \right)^{1/p'}$. Hence

$$\left( \sum_{i=1}^{n} \left( \int_{E_i} K(e, t) d\phi \right) \left( \phi(E_i)^{1/p} \right)^{p'} \right)^{1/p'} \leq \|T\| \left( \|\psi(e)\|^{1/q} \right)^{1/p'}$$

for all partitions $\{E_i\}$ of $R$ as described above and for any $e \in F(S)$, $0 < \psi(e) < \infty$. Hence if the supremum is taken over all such partitions.
\[
\sup_{T} \left\{ \sum_{i=1}^{n} \left( \int_{E_i} |K(e, t)\,d\phi|^p \right)^{1/p'} \right\}^{1/p'} = \sup_{T} \left\{ \sum_{i=1}^{n} \left( \frac{1}{\phi(E_i)} \int_{E_i} |K(e, t)\,d\phi|^p \right)^{1/p'} \right\}^{1/p'} \\
= p' \cdot \text{var}_R \left\{ \int_{R} |K(e, t)\,d\phi| \right\}^{1/p'} \\
= \left( \int_{R} |K(e, t)|^{p'\,d\phi} \right)^{1/p'} \\
\leq \|T\| \|\psi(e)\|^{(q-1)/q}
\]

and hence for any \(e \in \mathcal{F}(S)\) of finite nonzero measure,
\[
\frac{\left[ \int_{R} |K(e, t)|^{p'\,d\phi} \right]^q}{[\psi(e)]^{q-1}} \leq \|T\|^{p'q} [\psi(e)]^{(p'q - 1)(q-1)}
\]

Let \(\{e_j\}\) be any finite family of subsets of \(S\) of finite nonzero measure, \(j = 1, 2, \ldots, m\). Then
\[
\sum_{j=1}^{m} \left[ \int_{R} |K(e_j, t)|^{p'\,d\phi} \right]^q [\psi(e_j)]^{q-1} \leq \|T\|^{p'q} \left( \sum_{j=1}^{m} [\psi(e_j)]^{(p'q - 1)(q-1)} \right),
\]
\[
\left[ \sum_{j=1}^{m} \left[ \int_{R} |K(e_j, t)|^{p'\,d\phi} \right]^q \right]^{1/q} [\psi(e_j)]^{q-1} \leq \|T\|^{p'} \left( \sum_{j=1}^{m} [\psi(e_j)]^{(p'q - 1)(q-1)} \right)^{1/q}.
\]

If the least upper bound of the two quantities is taken with respect to all such families \(\{e_j\}\) of \(S\) then
\[
\sup_{T} \left\{ \sum_{j=1}^{m} \left[ \int_{R} |K(e_j, t)|^{p'\,d\phi} \right]^q \right\}^{1/q} \leq q \cdot \text{var}_T \left( \int_{R} |K(e, t)|^{p'\,d\phi} \right) \leq \|T\|^{p'} \sup_{T} \left( \sum_{j=1}^{m} [\psi(e_j)]^{(p'q - 1)(q-1)} \right)^{1/q}
\]

and hence
\[
\left( q \cdot \text{var}_T \int_{R} |K(e, t)|^{p'\,d\phi} \right)^{1/p'} / \sup_{T} \left( \sum_{j=1}^{m} [\psi(e_j)]^{(p'q - 1)(q-1)} \right)^{1/q} \leq \|T\|.
\]

If \(S\) has finite measure and \((p' - 1)(q - 1) \geq 1\), the supremum in the
denominator is finite and this implies that \( \int_R |K(e, t)|^{p'} d\phi \) has bounded \( q \)-variation. In particular, if the measure of \( S \) is less than one and \((p' - 1)(q - 1) \geq 1\), then for each partition \( \{e_j\} \), \( \sum_{j=1}^{m} [\psi(e_j)]^{(p'-1)(q-1)} \leq 1 \) and

\[
q \text{-var}_S \left( \int_R |K(e, t)|^{p'} d\phi \right) \]^{1/p'} \leq \|T\|.
\]

However, \((p' - 1)(q - 1) \geq 1\) implies \( q \geq p'/(p' - 1) = p \). Thus if \( q \geq p \) the above inequalities will hold. Hence if \( \psi(S) \leq 1, q \geq p \), the double inequality \( q \text{-var} \int_R |K(e, t)|^{p'} d\phi \]^{1/p'} \leq \|T\| \leq q \text{-var} \int_R |K(e, t)|^{p'} d\phi \]^{1/p'} \) will hold. Thus if the right member of the inequality is finite the integral represents a bounded linear operator between \( L^p \) and \( V^q \), and if the norm of \( T \) is finite, the left member of the inequality is finite. It is to be noted that since there is no guarantee that \( \int_R |K(e, t)|^{p'} d\phi \) is an additive set function, the theorem of Riesz on the equivalence of the \( q \)-variation to the \( q \)th integral norm will not apply and the \( q \)-variation cannot be replaced by an integral.

In case \( p = q \), then \((q - 1)(p' - 1) = 1\) and in this case

\[
\sup_{\varphi} \sum_{j=1}^{m} [\psi(e_j)]^{(p'-1)(q-1)} = \psi(S).
\]

Hence we have the

**Corollary.** If \( S \) has finite measure and \( p = q \), then

\[
\left( q \text{-var}_S \int_R |K(e, t)|^{p'} d\phi \right) \]^{1/q} \leq \|T\|.
\]

**References**


**University of Wisconsin**