ON UNITARY DILATIONS OF CONTRACTIONS

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Using some relatively deep facts about complex functions and spectral measures, B. Sz.-Nagy [2] has recently proved that to every contraction $A$ on a Hilbert space $H$ there corresponds a unitary operator $U$ on a larger Hilbert space $K$ so that $U^n$ is a dilation of $A^n$ for every positive integer $n$. In other words, if $\|A\| \leq 1$, then $K$ and $U$ can be found so that $PU^nP = A^nP$, where $P$ is the projection from $K$ onto $H$. The purpose of this note is to prove the same theorem by directly exhibiting the unitary operator $U$.

Write $K$ for the direct sum of countably many copies of $H$, indexed by the set of all integers. The operator $U$ will be exhibited as a matrix whose entries $U(i, j)$ are operators on $H$ ($i, j = 0, \pm 1, \pm 2, \cdots$). If $P$ denotes the projection from $K$ onto the zeroth coordinate space, then the fact that $PU^nP = A^nP$ will find its matricial expression in the assertion that the $(0, 0)$ entry of $U^n$ is $A^n$. Let $S$ and $T$ be the positive operators on $H$ defined by $S^2 = 1 - AA^*$ and $T^2 = 1 - A^*A$, where $1$ denotes the identity operator. The operators $U(i, j)$ are then defined as follows: $U(0, 0) = A$, $U(-1, 1) = -A^*$, $U(-1, 0) = T$, $U(0, 1) = S$, $U(i, i+1) = 1$ when $i < -1$ and when $i > 0$, and $U(i, j) = 0$ in all other cases.

An elementary computation shows that $U$ is unitary once it is proved that $SA = AT$. This was proved by Halmos [1] by noting that $S^2A = AT^2$ and approximating the square root function in the unit interval by polynomials. The fact that $U^n$ is a dilation of $A^n$ is evident since, except for $U(0, 0)$, all the nonzero entries of $U$ are above the main diagonal.

The possibility of such an explicit proof was suggested by P. R. Halmos who also contributed several useful comments.

REFERENCES


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