DECOMPOSITION OF A GROUP WITH A SINGLE DEFINING RELATION INTO A FREE PRODUCT[1]

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Let G be a group with generators a_ν, $\nu=1, \cdots, n$. An application of any automorphism A of the free group on the a_ν or, equivalently, of a sequence of T-transformations (defined below) maps G upon an isomorphic group G'. If G is defined by a set of prescribed relations for the a_ν, G' can be defined by transcribing the original relations in terms of the $A^{-1}a_\nu$. Even if G is defined by a single relation, it is not known how far the set of all groups with a single defining relation and isomorphic to a given one is determined by the transformations A. However, Grushko's theorem [2][2] implies that at least the decomposibility of G into a free product of two of its proper subgroups can be made obvious by applying a properly chosen A. We shall show that for a G with a single defining relation a result of J. H. C. Whitehead [1] provides a constructive method for finding A and some simple tests for the free indecomposability of G.

DEFINITIONS AND REMARKS. (1) *T-transformations*. By a T-transformation on the generators $a_1, \cdots, a_n$ of the free group $F = F(a_1, \cdots, a_n)$ we mean a mapping of the form:

$$Ta_k = a_k \text{ for some fixed } k, \qquad 1 \leqq k \leqq n,$$

$$Ta_i = a_i \text{ or } a_i a_k^{\epsilon} \text{ or } a_k^{-\epsilon} a_i \text{ or } a_k^{-\epsilon} a_i a_k^{\epsilon}, \qquad i \neq k,\ 1 \leqq i \leqq n.$$

The Greek superscripts denote either 1 or -1. The symbol a_k is referred to as the *distinguished* symbol for the given T-transformation. Whenever necessary, we shall indicate the distinguished symbol by writing T_{a_k} rather than T.

(2) *The symbol* TW. The symbol TW ($W = W(a_1, \cdots, a_n)$, T denotes a T-transformation on $a_1, \cdots, a_n$) denotes the word obtained by reducing $W(Ta_1, \cdots, Ta_n)$ (i.e., by deleting all $a_\nu a_\nu^{-1}$, $a_\nu^{-1}a_\nu$ in it).

(3) *The symbol* $L(W)$. If W is a reduced word, then $L(W)$ denotes the number of symbols in W. We refer to this number as the *length* of W.

(4) *T-reductions and level transformations*. A T-transformation, as

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[2] Numbers in brackets refer to the references at the end of the paper.

applied to a (reduced) word W, is called:

$$\text{a } T\text{-reduction if } L(TW) < L(W)$$

and

$$\text{a level transformation if } L(TW) = L(W).$$

(5) *Internal and external T-transformations.* Let W be a reduced word, $W=W(a_1, \cdots, a_n)$. Regard it as a word in the symbols $a_1, \cdots, a_n, a$. If $T=T_{a_i}$, then T is called an *internal* T-transformation with respect to W. If $T=T_a$, then T is called an *external* T-transformation with respect to W.

(6) *Active and inactive symbols; right, left, and transform symbols. Active and inactive words.* Consider a T-transformation on the symbols $a_1, \cdots, a_n$. We call a_i *inactive* if $Ta_i=a_i$. We call a_i *active* if $Ta_i\neq a_i$. If a_i^ρ is an active symbol and $Ta_i^\rho=a_i^\rho a_k^\epsilon$ $(T=T_{a_k})$, we call a_i^ρ a *right* symbol. If $Ta_i^\rho=a_k^{-\epsilon}a_i^\rho$, we call a_i^ρ a *left* symbol. If $Ta_i=a_k^{-\epsilon}a_ia_k^\epsilon$, we call a_i a *transform* (symbol). A word W is said to be *active* (*inactive*) T if one (none) of its symbols is active T.

(7) *Conjugate T-transformations.* Consider $W=W(a_1, \cdots, a_n)$ and let $a\neq a_i$, $1\leqq i\leqq n$. Let T_a be a definite T-transformation on the symbols $a_1, \cdots, a_n, a$. We shall call an internal T-transformation T_{a_k} on the symbols $a_1, \cdots, a_n$ *conjugate* to T_a if, for $a_i\neq a_k$,

$$T_aa_i^\rho = a_i^\rho a^\epsilon \text{ implies } T_{a_k}a_i^\rho = a_i^\rho a_k^\epsilon,$$

$$T_aa_i = a^{-\epsilon}a_ia^\epsilon \text{ implies } T_{a_k}a_i = a_k^{-\epsilon}a_ia_k^\epsilon,$$

$$T_aa_i = a_i \text{ implies } T_{a_k}a_i = a_i.$$

(8) *Disjoint words.* Two words are said to be *disjoint* if the symbols which occur in one of them do not occur in the other (a and a^{-1} are *not* disjoint).

(9) *Minimal words.* $W=W(a_1, \cdots, a_n)$ is said to be *minimal* (T) or, simply, *minimal*, if $L(TW)\geqq L(W)$ for every T on $a_1, \cdots, a_n$. If W is minimal with respect to all T-transformations on $a_1, \cdots, a_n$, then it is also minimal with respect to all T-transformations on a set of symbols containing the symbols $a_1, \cdots, a_n$.

(10) *Use of the term "involves."* If it is impossible to eliminate a symbol a appearing in a word W by writing W cyclically and deleting all pairs $(a_\nu a_\nu^{-1})^{\pm 1}$, we say that W *involves* a.

LEMMA. *Let $W=W(a_1, \cdots, a_n)$ be a (reduced) minimal word in $a_1, \cdots, a_n$. Assume that W is nontrivial, i.e.,* $L(W)>1$. Let $a\neq a_i$,

$i=1, \cdots, n$. Let $T_a a_i \neq a_i$ for at least one a_i in W. Then $L(T_a W) - L(W) \geqq 2$.

Obviously the to-be-proved increase in length of W under T is due to "trapped" a-symbols.

PROOF. Note that if $L(W)>1$ and W is minimal, it must contain at least two symbols of a kind, if any. For, let us assume that W contains a single symbol a_1 and $W= \cdots a_1 a_j \cdots, j\neq 1$. Then the T-transformation: $a_1 \rightarrow a_1 a_j^{-1}$, $a_i \rightarrow a_i$, $i\neq 1$, decreases $L(W)$ by 1, which contradicts the assumed minimality of W.

Now consider a definite T_a such that $L(T_a W)=L(W)$. It is clear that $T_a W = W$. Also, W must be a product of the form:

$$W = \prod [(i\text{'s or } 1) \text{ an } r \ (t\text{'s or } 1) \text{ an } l \ (i\text{'s or } 1)], \tag{1}$$

where i = inactive symbol, r = right symbol, l = left symbol, t = transform. Let a_1 be the first right symbol in the above product. It is not difficult to see that the conjugate T_{a_1} of T_a (see definition (7)) applied to W would result in the elimination of all a_1 symbols from W without insertion of any other symbols. But this would decrease $L(W)$ which is impossible in view of the assumed minimality of W.

We know by now that $L(T_a W) - L(W) \geqq 1$. The "trapping" of an a-symbol in $T_a W$ may be effected by a right a_i, a left a_i, or a transform a_i. We know that W must contain at least two such a_i symbols. We claim that each of these a_i symbols "traps" an a-symbol. We assume that this statement is false and proceed to deduce a contradiction.

We observe that every active (under T_a) symbol a_q in W which does not "trap" an a-symbol must be contained in a "block" of the form:

$$[\text{right symbol (transforms or 1) left symbol}].$$

As for the "trapping" symbol a_i^ρ we assume, at first, that it is a right or a left symbol under T_a. Then, $W = W_1 a_i^\rho W_2$, where $W_i = 1$ or a word of the form (1) above and not both $W_i = 1$. As before, T_{a_i}, assumed to be conjugate to T_a, applied to W will eliminate all a_i symbols in W other than a_i^ρ and will not introduce any new symbols in place of the eliminated symbols. This would decrease $L(W)$ by at least 1, which is impossible.

There remains the possibility that the trapping symbol a_i^ρ is a transform under T_a. Then

$$W = W_1[a_i^\rho \text{ (transforms or 1) left symbol}]\, W_2 = W_1 A W_2 \tag{2}$$

or

(3) $W = W_1$ [right symbol (transforms or 1) $a_i^\rho]W_2 = W_1\bar{A}W_2$.

We again emphasize the fact that not both words W_1 and W_2 can be 1, for the left (right) symbol in (2) ((3)) must have a counterpart. In (2), $T_{a_i}^\nu$, where T_{a_i} is assumed to be conjugate to T_a and the value of $\nu=\pm 1$ is determined by the equation: $T_{a_i}^\nu l=a_i^{-\rho}l$, l=left symbol, eliminates an a_i^ρ in A when applied to $W=W_1AW_2$. Also, $T_{a_i}^\nu W_j=W_j$, $j=1, 2$. Similarly, in (3), $T_{a_i}^\nu$, where the value of ν is determined by the equation: $T_{a_i}^\nu r=ra_i^{-\rho}$, r=right symbol, eliminates an a_i^ρ in $\bar{A}$ when applied to $W=W_1\bar{A}W_2$. Also, $T_{a_i}^\nu W_j=W_j, j=1, 2$. Thus, in both cases $L(W)$ is decreased which is impossible in view of the assumed minimality of W.

We now state a fundamental theorem of Whitehead (Theorem 3 in [1]): "Any two equivalent minimal sets (T) are interchangeable by level T-transformations."

It follows immediately from this result that if W_1 and W_2 are two minimal forms of a word $W=W(a_1, \cdots, a_n)$ obtained from W by means of T-transformations, then $L(W_1)=L(W_2)$. This fact and our lemma permit us to prove the following

COROLLARY. *Let $W=W(a_1, \cdots, a_n)$. Let W_1 and W_2 be two minimal forms of W. Then W_1 and W_2 contain the same number of distinct symbols.*

PROOF. Note that if T is a level transformation with respect to a minimal word $V=V(a_1, \cdots, a_n)$ which is active T, then:

(a): TV is minimal (by Whitehead's theorem above);

(b): T is internal with respect to V (if V is trivial, i.e. $L(V)=1$, this statement is obvious; if V is nontrivial the statement follows from our lemma);

(c): The number of distinct symbols in V equals the number of distinct symbols in TV (since T is both level and internal).

Our corollary is trivial if $L(W_1)=L(W_2)=1$. We may therefore assume that $L(W_1)=L(W_2)>1$. By Whitehead's theorem there exists a (finite) chain of level T-transformations $T_1, \cdots, T_k$ such that $T_1 \cdots T_kW_1=W_2$, where $T_{i+1} \cdots T_kW_1$ may be supposed active T_i. The desired conclusion now follows immediately by induction on k using the observations (a), (b), (c) in the beginning of the proof.

It follows from Grushko's theorem (cf. [2]) that: A group with $n\geqq 2$ generators and a single defining relation involving the n generators can be decomposed into a free product if and only if it is possible to reduce the number of distinct generators in the left side of the defining relation by means of a suitable free automorphism on the generators.

This result and the corollary to our lemma permit us to prove

THEOREM 1. *Let* $G=G[a_1, \cdots, a_n; R(a_1, \cdots, a_n)=1]$, *where all* a_i *are involved in* R. *Let* H *be the free product of an infinite cyclic group* $\{a\}$ *and a nontrivial group* B *with generators* $b_\nu \neq a$. *Then* $G \simeq H$ *if and only if any minimal form of* R *contains at most* $n-1$ *distinct* a_i's.

PROOF. The sufficiency part of the proof is obvious. To prove the necessity of our condition we assume that $G \simeq H$ and that some minimal form of R contains n distinct symbols. By the corollary to our lemma every minimal form of R contains n distinct symbols. On the other hand, it follows from Grushko's theorem that it is possible, by applying a suitable free automorphism to the generators of G, to find a representation of G such that the word on the left side of the defining relation associated with this representation contains at most $n-1$ symbols. Minimizing this word we obtain a minimal form of R containing at most $n-1$ symbols, which contradicts the corollary to our lemma.

REMARK. If the number of generators of G exceeds the number of generators involved in R, G is obviously representable as a free product of the required form.

It is clear that if the left side of the defining relation associated with a certain set of generators of G is minimal, then G cannot be represented as a free product. We now state and prove criteria which ensure the minimality of a word $W(a_1, \cdots, a_n)$ and so the indecomposability into a free product of $G=G[a_1, \cdots, a_n; W(a_1, \cdots, a_n) =1]$.

THEOREM 2. *For a product of disjoint minimal words (see definitions* (8) *and* (9)) *to be minimal it is necessary and sufficient that each factor* W_p *of the product be nontrivial (i.e.,* $L(W_p)>1$ *for each* W_p).

PROOF. Let $W=W_1 \cdots W_m$, W_p minimal and nontrivial, W_p, W_q disjoint for $p\neq q$, $1\leqq p,\ q\leqq m$. Consider any T_a, a^ρ in W_i. Then $L(T_aW_i)-L(W_i)\geqq 0$. Also, if $j\neq i$, (i) $L(T_aW_j)-L(W_j)=0$ if W_j does not contain symbols active T_a, (ii) $L(T_aW_j)-L(W_j)\geqq 2$ if W_j contains symbols active T_a. In case (i) no deletions can take place at the junction(s) between $T_aW_j=W_j$ and its neighbor(s) T_aW_k and its length remains fixed. In case (ii) the length of W_j increases as a result of the application of T_a by at least 2 and its losses, resulting from deletions at the junction(s) between T_aW_j and its neighbor(s), cannot exceed 1 if $j=1$ or m, and they cannot exceed 2 if $1<j<m$. Now, $W=AW_iB$, a^ρ in W_i, A and B not both 1. Assume $A\neq 1$. If all the words in A are inactive with respect to T_a, then $L(T_aA)-L(A)=0$, and no deletions take place between A and T_aW_i. On the other hand,

if at least one word in A is active T_a, then $L(T_aA)-L(A)\geqq 2$. Similarly for B. Now consider $T_aA\cdot T_aW_i\cdot T_aB$. If no deletions take place at either junction, $L(T_aW)-L(W)\geqq 0$. If deletion takes place at the first junction, say, then the last word in A must be active T_a and $L(T_aA)-L(A)\geqq 2$. It is by now obvious that in any case $L(T_aW)-L(W)\geqq 0$, q.e.d.

As an immediate application of Theorem 2 we have:

The fundamental group of a closed surface cannot be represented as a free product.

THEOREM 3. *Let all exponents in $W(a_1, \cdots, a_n)$ be $\geqq 2$. Then W is minimal.*

PROOF. Apply a definite $T=T_{a_1}$, say, to W, which we can write as

$$W = W_1(a_2, \cdots, a_n)a_1^{k_1}W_2(a_2, \cdots, a_n)a_1^{k_2}\cdots$$

$$\cdot W_p(a_2, \cdots, a_n)a_1^{k_p}W_{p+1}(a_2, \cdots, a_n)$$

(where W_1 or W_{p+1} or both may be 1). Then

$$T_{a_1}W = (T_{a_1}W_1)a_1^{k_1}(T_{a_1}W_2)a_1^{k_2}\cdots(T_{a_1}W_p)a_1^{k_p}(T_{a_1}W_{p+1}).$$

Observe that deletions, if any, can take place only at *one* end of $T_{a_1}W_j$ (cf. the definition of a T-transformation). This fact and our lemma yield immediately the desired conclusion.

It is obvious that the theorem holds in the following slightly more general form:

Let all exponents associated with a generator a_i in W be of the same sign and in absolute value $\geqq 2$. Then W is minimal.

THEOREM 4. *Let $W=V^m$, V minimal. Then W is minimal.*

PROOF. The result is trivial for $m=1$. Let $m=2$. If T is a definite T-transformation, then $T(V^2)=(TV)(TV)$ and deletion can take place between the two bracketed words if and only if TV is a transform, in which case $L(TV)-L(V)\geqq 1$. Consequently $L[(TV)^2]-L(V^2)\geqq 0$, i.e., V^2 is minimal. It follows by induction on k that V^{2k}, $k=$a positive integer, is minimal. Using the reasoning employed in proving the case $m=2$, we can prove our result for V^{2k+1}, and so for V^m.

The following theorem can be easily proved:

THEOREM 5. *Let $K=K[a_1, a_2; R(a_1, a_2)=1]$ and let $G=G[a, b; b^n=1]$.*

Then, for $G \simeq K$ it is necessary and sufficient that R be cyclically equivalent to $A^n(a_1, a_2)$ where $A(a_1, a_2)$ is a primitive element in the free group $F(a_1, a_2)$.

REFERENCES

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A THEOREM ON COMMUTATIVE POWER ASSOCIATIVE LOOP ALGEBRAS[1]

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Let L be a loop, written multiplicatively, and F an arbitrary field. Define multiplication in the vector space A, of all formal sums of a finite number of elements in L with coefficients in F, by the use of both distributive laws and the definition of multiplication in L. The resulting *loop algebra* $A(L)$ over F is a linear nonassociative algebra (associative, if and only if L is a group).

An algebra A is said to be power associative if the subalgebra $F[x]$ generated by an element x is an associative algebra for every x of A.

THEOREM. *Let $A(L)$ be a loop algebra over a field of characteristic not 2. A necessary and sufficient condition that $A(L)$ be a commutative, power associative algebra is that L be a commutative group.*

PROOF. Assume that $A(L)$ is a commutative, power associative algebra. Clearly L must be commutative and $x^2 \cdot x^2 = (x^2 \cdot x) \cdot x$ for all x of $A(L)$. Under the hypothesis that the characteristic of F is not 2, a linearization[2] of this power identity yields

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[2] See A. A. Albert, *On the power associativity of rings*, Summa Brasiliensis Mathematicae vol. 2, no. 2, pp. 21–32.