A REMARK ON FINITELY GENERATED
NILPOTENT GROUPS

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In this note we use the word nilpotent in the strong sense; a group $G$ is nilpotent if its lower central series, defined by $H_0 = G$ and $H_{i+1} = (H_i, G)$, terminates in the identity in a finite number $c$ of steps, and $c$ is then the class of $G$. As usual, $G^n$ denotes the subgroup of $G$ generated by the $n$th powers of elements of $G$. Our object is to prove the following result.

Theorem 1. If $G$ is a finitely generated nilpotent group, then the intersection of the groups $G^p$, for any infinite set of primes $p$, is finite.

We recall first some well known facts about finitely generated nilpotent groups. They will be found, in essence, in Hall [2] and Hirsch [3]. First, if $X$ is a subset of $G$ that generates $G$ modulo its derived group $H_1$, then $H_{c-1}$ is generated modulo $H_i$ by the left normed commutators

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(\cdots ((x_1, x_2), x_3), \cdots , x_i), \quad x_r \in X.
$$

Thus $H_{c-1}$ is finitely generated modulo $H_i$. In particular, $H_{c-1}$ is a finitely generated abelian group, and its subgroups therefore satisfy the maximal condition. By induction on $c$, so do those of $G$. Again, if any of $x_r$ is of finite order modulo $H_i$, then the commutator (1) is of finite order modulo $H_i$. Hence finitely many elements of $G$ of finite order generate a finite subgroup, and it follows from the maximal condition on the subgroups of $G$ that in fact there are only a finite number of elements of finite order in $G$. In particular, there are only a finite number of primes $p$ for which $G$ contains elements of order $p$.

Lastly, for a prime $q$ greater than the class $c$ of $G$, Hall's formula becomes

$$(xy)^q = x^q y^q z_1^q \cdots z_r^q,$$

where each $z_j$ is a commutator in $x$ and $y$. It follows easily, by backward induction on the weight of $y$ (the least integer $i$ such that $y$ does not belong to $H_i$), that for some $u$ in $G$, $x^q y^q = u^q$. That is, $G^q$ is not merely generated by, but consists of the $q$th powers of elements of $G$.

We can now prove Theorem 1. If the class $c$ of $G$ is 1, so that $G$ is abelian, the theorem follows immediately from the basis theorem for finitely generated abelian groups. We therefore use induction on $c$, and

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assume $c > 1$. Let $K$ be the intersection of an infinite set of subgroups $G^p$. By the hypothesis of the induction, $KH_{c-1}/H_{c-1}$ is finite, and hence so is the isomorphic group $K/(K \cap H_{c-1})$. It is sufficient therefore to prove that $K \cap H_{c-1}$ is finite. If $y \in K$, then surely $y \in G^p$ for an infinity of primes $q$ with $q > c$; thus for an infinity of $q$ there exists $y_q$ such that $y_q^q = y$. If also $y \in H_{c-1}$, the cyclic group $\{y\}$ is normal, and unless $y_q \in \{y\}$, the factor group $G/\{y\}$ contains an element of order $q$. Since this is possible for only a finite set of primes $q$, for some $q$, $y_q \in \{y\}$, whence $y$ is of finite order. That is, $K \cap H_{c-1}$ contains only elements of finite order, and is therefore a finite group. This concludes the proof.

It is perhaps of interest to remark that Theorem 1 yields a short proof of the following theorem of Baer [1].

**Theorem 2.** There is an integer $n$ such that the intersection of all characteristic subgroups of $G$ whose indices are prime powers $p^a$ with $a \leq n$ is the identity.

For, by a theorem of Hirsch [4], $G$ (or, indeed, any soluble group with the maximal condition for subgroups) has a subgroup $N$ of finite index which contains no elements of finite order. We can take this subgroup to be characteristic. For, if its index is $h$, we can replace it by the intersection of all subgroups of $G$ of index $h$, which (Baer, loc. cit.) is still of finite index. Then $G/N$ is a finite nilpotent group, and so the direct product of its Sylow subgroups, whence $N$ is the intersection of a finite set of characteristic subgroups of $G$, whose indices are prime powers. If to these we add any infinite set of subgroups $G^p$, we obtain a set whose intersection is the identity. For this intersection contains no element of finite order by choice of $N$, and none of infinite order by Theorem 1. All the groups are characteristic subgroups of $G$ of prime power index $p^a$, and to show that the exponents $a$ are bounded, we may concentrate on the groups $G^p$, since the others are a finite set only. But if the number of generators of $H_{i-1}/H_i$ is $r(i)$, the index of $G^p$ is $p^a$ with $a \leq r(1) + r(2) + \cdots + r(c).

**References**


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