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## A NOTE ON HAAR MEASURE<sup>1</sup>

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We prove a direct sum theorem for Haar measure which is a generalization, to arbitrary locally compact topological groups, of a theorem for Lie groups due to Mostow [3]. It is well known that the Haar integral on a *compact* group may be obtained by a simple averaging process. On a *vector* group, the Haar integral is essentially the Lebesgue integral. On a *discrete* group it is summation. Our theorem asserts that the formation of a Haar integral on any locally compact group must be a composite of these three processes. Hence no locally compact group exists for which the Haar integral must be constructed in an essentially more novel manner.

**THEOREM.** *Given any locally compact topological group  $G$ , there exists a compact subgroup  $K$ , a subspace  $E$  which is homeomorphic to an  $n$ -dimensional Euclidean space, and a discrete subset  $D$ , such that the mapping  $\theta(a, b, c) = abc$  is a homeomorphism of the Cartesian product  $D \times E \times K$  with  $G$ , and carries the product measure  $m_D \times m_E \times m_K$  over to  $m_G$ . Here  $m_D$  is discrete measure in  $D$ ,  $m_E$  is Lebesgue measure in  $E$  relative to suitable coordinates,  $m_K$  is Haar measure in  $K$ , and  $m_G$  is left Haar measure in  $G$ . (Note that  $E$  need not be a subgroup.)*

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We first prove this theorem for a group  $G$  having a compact normal subgroup  $N$  such that  $G/N$  is a connected Lie group, making use of the fact that Mostow has proved it for Lie groups. Then we use the fact, proved by Yamabe, that any locally compact group has an open subgroup with this property. We must first prove the following lemma, which was proved for connected groups by Iwasawa [2, p. 545].

LEMMA 1. *Let  $G$  be a locally compact group and  $N$  a compact normal subgroup of  $G$ . If  $H_1$  is a closed subgroup of  $G$  containing  $N$  and if  $H_1/N$  is isomorphic with the reals, then there is contained in the connected component of the identity of  $G$  a closed subgroup  $H$  also isomorphic with the reals such that  $H_1 = HN$ ,  $H \cap N = \{e\}$ .*

PROOF. It suffices to prove the assertion for the case  $G = H_1$ . Let  $G^0$  be the connected component of the identity of  $G$ . The natural homomorphism  $\pi$  of  $G$  onto  $G/N$  maps  $G^0$  onto  $G^0N/N$ . Since  $N$  is compact,  $G^0N/N$  is topologically isomorphic with  $G^0/(G^0 \cap N)$  (see [4, p. 17]), so we have a natural homomorphism  $\pi'$  of  $G^0$  onto  $G^0/(G^0 \cap N)$ .

Since  $G^0$  is connected and  $\pi$  is continuous, the image of  $G^0$  under  $\pi$  is a connected subgroup of  $G/N$ . Since  $G/N$  is isomorphic with the reals, this image is either  $\{e\}$  or a subgroup isomorphic with the reals. The first of these possibilities gives  $G^0/(G^0 \cap N) \approx \{e\}$ , implying  $G^0 \subseteq N$ . Then  $G/N \approx (G/G^0)/(N/G^0)$  (see [4, p. 12]) and this is totally disconnected because  $G/G^0$  is. This contradicts the isomorphism of  $G/N$  with the reals. So we can have only the second possibility, that  $G^0/(G^0 \cap N)$  is isomorphic with the reals. In this case, Lemma 4.7 of [2] applies directly to give the result since  $G^0$  is connected.

LEMMA 2. *Let  $G$  be a locally compact group which has a compact normal subgroup  $N$  such that  $G/N$  is a connected Lie group. Then the assertion of the theorem holds with  $D = \{e\}$ .*

PROOF. Theorem 1 of [3] implies that, given any connected Lie group  $G'$ , there exists a compact subgroup  $K'$  and a Euclidean subspace  $E'$  such that the mapping  $\theta$  of  $E' \times K'$  onto  $G'$  defined by  $\theta(a, b) = ab$  is a homeomorphism; also,  $E' = U'_1 U'_2 \cdots U'_n$  where the  $U'_i$  are isomorphic with the reals and this decomposition is topological in the sense that the mapping  $\phi(u_1, u_2, \cdots, u_n) = u_1 u_2 \cdots u_n$  of  $U'_1 \times U'_2 \times \cdots \times U'_n$  onto  $E'$  is a homeomorphism. Moreover, given any continuous function  $f$  which vanishes outside a compact set,

$$\int_{G'} f(x) dm_{G'}(x) = \int_{E'} dm_{E'} \int_{K'} f(ab) dm_{K'}(b)$$

where  $m_{E'}$  is Lebesgue measure relative to the coordinates  $\phi^{-1}$  and  $m_{K'}$  is Haar measure in  $K'$ .

Letting  $\pi$  denote the natural homomorphism of  $G$  onto  $G/N$ , and applying Lemma 1, we find subgroups  $U_1, U_2, \dots, U_n$ , each isomorphic with the reals, such that  $\pi^{-1}(U_i) = U_i N$ ,  $U_i \cap N = \{e\}$ . Denoting  $\pi^{-1}(K')$  by  $K$ , we have  $G = U_1 U_2 \dots U_n K$  where  $K/N = K'$ ,  $U_i \cap K = \{e\}$ ,  $i = 1, 2, \dots, n$ . Using arguments essentially like those in [2], it is seen easily that this decomposition is topological; in particular,  $E = U_1 U_2 \dots U_n$  is homeomorphic via  $\pi$  with  $E'$ .

Let  $f$  be any continuous function on  $G$  which vanishes outside a compact set. By a theorem of Weil [4, p. 45],

$$\int_G f dm_G = \int_{G/N} dm_{G/N} \int_N f(xn) dm_N(n)$$

where  $m_{G/N}$  and  $m_N$  are left Haar measures in  $G/N$  and  $N$  respectively. Observing that  $\int_N f(xn) dm_N(n)$  is constant on cosets of  $N$  and hence defines a function  $g(x)$  on  $G' = G/N$  which vanishes outside a compact set, we apply Mostow's theorem to give

$$\int_G f dm_G = \int_{E'} dm_{E'}(a') \int_{K'} g(a'b) dm_{K'}(b).$$

Applying Weil's theorem again, this time to  $K$ , we have, for any  $a$  in  $E$ ,

$$\int_K f(ak) dm_K(k) = \int_{K'} dm_{K'} \int_N f(acn) dm_N(n) = \int_{K'} g(a'b) dm_{K'}(b)$$

where  $a' = \pi a$ . So if we carry  $m_{E'}$  back to  $E$  via the inverse of the restriction of  $\pi$  to  $E$ , we have a Lebesgue measure  $m_E$  in  $E$  and can write

$$\int_G f dm_G = \int_E dm_E(a) \int_K f(ak) dm_K(k).$$

We have proved this formula only for functions which are continuous and have compact support, but Ambrose shows in [1] how to extend such formulas to a wider class of functions and obtain the asserted measure-theoretic fact when the Haar measures are taken to be nontrivial left-invariant Baire measures. His direct sum theorem is stated only for invariant subgroups, but nevertheless his proofs carry over to our case with little modification.

PROOF OF THEOREM. Yamabe showed in [5] that any locally compact topological group contains an open subgroup  $G^0$  and a compact

normal subgroup  $N \subseteq G^0$  such that  $G^0/N$  is a connected Lie group. Choose representing elements from the left cosets of  $G^0$  in  $G$  to make up a set  $D$ , so that  $G = DG^0$ . Then  $D$  is discrete in the induced topology and for integrable  $f$ ,

$$\int f dm_G = \sum_{d \in D} \int_{G^0} f(dx) dm_{G^0}(x)$$

so the homeomorphism  $\theta(d, x) = dx$  of  $D \times G^0$  onto  $G$  carries  $m_D \times m_{G^0}$  onto  $m_G$ . This, together with Lemma 2, proves the theorem.

We remark in concluding that we can prove Mostow's theorem, in the weakened form in which we used it here, more directly by applying methods due to Weil [4]. We intend to discuss this, together with some applications of the theorem proved here to the theory of group representations, in a later paper.

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