A SPECIAL DETERMINANT

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Let p be a prime and let $R(r)$ denote the least non-negative residue of $r \pmod p$. The properties of Maillet's determinant (for references see [2]) $D_p = |R(rs')|$ $(r,\ s = 1, \cdots, (p-1)/2)$ where $ss' \equiv 1 \pmod p$, suggest that it may be of interest to discuss the determinant

$$\Delta_k = \left| R((r-s)^k) \right| \qquad (r, s = 0, \cdots, p-1; 1 \leqq k \leqq p-1). \tag{1}$$

Clearly Δ_k is a circulant. Consequently

$$\Delta_k = \sum_{r=1}^{p-1} R(r^k) \cdot \prod_{s=1}^{p-1} \sum_{r=1}^{p-1} R(r^k)\epsilon^{rs}, \qquad \text{where } \epsilon = e^{2\pi i/p}. \tag{2}$$

Now by the binomial theorem

$$\begin{aligned} (1-\epsilon)^{p-1-k} &\equiv \sum_{r=0}^{p-1-k} \frac{(k+1)\cdots(k+r)}{r!}\epsilon^r \\ &\equiv \sum_{r=0}^{p-1-k} \frac{(r+1)\cdots(r+k)}{k!}\epsilon^r \qquad \pmod p. \end{aligned} \tag{3}$$

Next recall that (see for example [3, p. 207])

$$x^k = \sum_{s=1}^{k} a_{ks}\binom{x+s-1}{k} \qquad (a_{ks} = a_{k,k-s+1},\ k \geqq 1), \tag{4}$$

where the a_{ks} (Eulerian coefficients) are positive integers; clearly (4) implies

$$\sum_{s=1}^{k} a_{ks} = k!. \tag{5}$$

Then using (3) and (4) we get

$$\begin{aligned} (1-\epsilon)^{p-1-k}\sum_{s=1}^{k} a_{ks}\epsilon^{s-1} &\equiv \sum_{r,s}\binom{r+k}{k} a_{ks}\epsilon^{r+s-1} \\ &\equiv \sum_{t=1}^{p-1}\epsilon^{t-1}\sum_{s=1}^{k} a_{ks}\binom{t-s+k}{k} \\ &\equiv \sum_{t=1}^{p-1} t^k\epsilon^{t-1} \qquad \pmod p. \end{aligned} \tag{6}$$

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Also it is clear from (5) that

$$\sum_{s=1}^{k} a_{ks}\epsilon^{s-1} \equiv \sum_{s=1}^{k} a_{ks} \equiv k! \qquad (\text{mod } 1-\epsilon). \tag{7}$$

Thus it follows from (6) and (7) that the number

$$\alpha = \sum_{t=1}^{p-1} t^k \epsilon^{t-1}$$

is divisible by $(1-\epsilon)^{p-1-k}$ and not by $(1-\epsilon)^{p-k}$. We recall that in the cyclotomic field generated by ϵ we have the prime ideal factorization

$$(p) = (1-\epsilon)^{p-1}.$$

Now in the double product in the right member of (2), the sum

$$\sum_{r=1}^{p-1} R(r^k)\epsilon^{rs} \equiv \epsilon^s \sum_{r=1}^{p-1} r^k \epsilon^{s(r-1)} \qquad (\text{mod } p)$$

and is therefore divisible by exactly $(1-\epsilon)^{p-1-k}$. More precisely by (7)

$$\sum_{r=1}^{p-1} R(r^k)\epsilon^{rs} \equiv \epsilon^s k!\,(1-\epsilon)^{p-1-k} \qquad (\text{mod } (1-\epsilon)^{p-k})$$

and therefore

$$(1-\epsilon)^{-p+1+k} \sum_{r=1}^{p-1} R(r^k)\epsilon^{rs} \equiv \epsilon^s k! \qquad (\text{mod } 1-\epsilon).$$

Multiplying together these congruences we get

$$p^{-p+1+k} \prod_{s=1}^{p-1} \sum_{r=1}^{p-1} R(r^k)\epsilon^{rs} \equiv (k!)^{p-1} \equiv 1 \qquad (\text{mod } 1-\epsilon). \tag{8}$$

Since the left number is a rational integer, (8) holds (mod p). Thus substituting in (2) it is clear that for $k<p-1$

$$\Delta_k \equiv p^{p-1-k} \sum_{r=1}^{p-1} R(r^k) \qquad (\text{mod } p^{p-k+1}). \tag{9}$$

Put

$$S_k = \sum_{r=1}^{p-1} R(r^k);$$

since $p \mid S_k$, $p^2 \nmid S_k$ for $1 \leqq k < p-1$, it follows from (9) that

$$p^{p-k} \mid \Delta_k, \; p^{p-k+1} \nmid \Delta_k \qquad (1 \leqq k < p-1). \tag{10}$$

To get a more precise result, note first that if $a=(k, p-1)$, then $S_k=S_a$. Put $p-1=ab$; then in the first place

$$S_a = p(p-1)/2 \qquad (b \text{ even}). \tag{11}$$

For b odd, on the other hand, we may prove by the method used in [1] that

$$S_a \equiv -\frac{p}{2} + p\sum_u \frac{B_{bu+1}}{bu+1} \pmod{p^2}, \tag{12}$$

where B_m denotes a Bernoulli number in the even suffix notation, and the summation is over $u=1, 3, \cdots, a-1$.

We remark that for $k=p-1$ we have the easily verified formula

$$\Delta_{p-1} = \left| 1 - \delta_{rs} \right| = p - 1,$$

where δ_{rs} is the Kronecker delta. Also for $k=1$ we have the exact result

$$\Delta_1 = (p-1)p^{p-1}/2.$$

It follows from (2) and (8) that Δ_k never vanishes.

References

1. L. Carlitz, *The first factor of the class number of a cyclic field*, Canadian Journal of Mathematics vol. 6 (1954) pp. 23–26.

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3. J. Worpitzky, *Studien über die Bernoullischen und Eulerschen Zahlen*, J. Reine Angew. Math. vol. 94 (1883) pp. 202–322.

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