

POSITIVE FUNCTIONS ON C^* -ALGEBRAS

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1. **Introduction.** Let X be any set, let \mathfrak{S} be a Boolean σ -algebra of subsets of X , and let F be a function from \mathfrak{S} to non-negative operators on a Hilbert space \mathfrak{H} such that $F(X) = 1$ and F is countably-additive in the weak operator topology. Neumark [2] has shown that there exists a Hilbert space K of which \mathfrak{H} is a subspace and a spectral measure E defined on \mathfrak{S} such that $F(S)P = PE(S)P$ for all S in \mathfrak{S} , where P is projection of K on \mathfrak{H} . Let us rephrase this situation so that we speak of algebras rather than Boolean algebras and linear functions rather than measures. Thus, we consider, instead of the Boolean σ -algebra \mathfrak{S} , the C^* -algebra \mathcal{A} of all bounded functions on X which are measurable with respect to \mathfrak{S} . A C^* -algebra is defined as a complex Banach algebra with an involution $x \rightarrow x^*$ such that $\|xx^*\| = \|x\|^2$ for all x in the algebra. The measure F is supplanted by the linear function μ on \mathcal{A}

$$\mu(f) = \int f(\gamma) dF(\gamma), \quad f \in \mathcal{A},$$

where the integral is to be taken in the weak sense. The theorem now asserts that $\mu(f)P = P\rho(f)P$, where

$$\rho(f) = \int f(\gamma) dE(\gamma), \quad f \in \mathcal{A}.$$

In the original formulation, E was an improvement over F because E was a *spectral* measure; in the reformulation, ρ is an improvement over μ since ρ is a $*$ -homomorphism. When the situation is phrased in this manner, the question naturally occurs: "Is it essential that the algebra \mathcal{A} be commutative?" The present paper is devoted to a discussion of this point.

2. **The main theorem.** If \mathcal{A} and \mathcal{B} are C^* -algebras and μ is a linear function from \mathcal{A} to \mathcal{B} , we shall say that μ is *positive* if $\mu(A) \geq 0$ whenever $A \in \mathcal{A}$ and $A \geq 0$. The algebra of $n \times n$ matrices with entries in \mathcal{A} is also a C^* -algebra, which we shall denote by $\mathcal{A}^{(n)}$. By applying μ to each entry of an element of $\mathcal{A}^{(n)}$, we obtain an element of $\mathcal{B}^{(n)}$; this linear function from $\mathcal{A}^{(n)}$ to $\mathcal{B}^{(n)}$ will be denoted by $\mu^{(n)}$. We shall say that μ is *completely positive* if $\mu^{(n)}$ is positive for each positive integer n .

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THEOREM 1. *Let \mathcal{A} be a C^* -algebra with a unit, let \mathfrak{H} be a Hilbert space, and let μ be a linear function from \mathcal{A} to operators on \mathfrak{H} . Then a necessary and sufficient condition that μ have the form*

$$\mu(A) = V^* \rho(A) V \quad \text{for all } A \in \mathcal{A},$$

where V is a bounded linear transformation from \mathfrak{H} to a Hilbert space \mathfrak{K} and ρ is a $*$ -representation of \mathcal{A} into operators on \mathfrak{K} , is that μ be completely positive.

PROOF OF NECESSITY. Suppose that $\mu(A) = V^* \rho(A) V$. Let $M = (A_{ij})$ be a non-negative matrix in $\mathcal{A}^{(n)}$. Then $\mu^{(n)}(M)$ is an operator on the direct sum of \mathfrak{H} with itself n times. What we have to check is that

$$\sum_{i,j} (\mu(A_{ij}) x_j, x_i) \geq 0$$

whenever x_1, \dots, x_n are vectors in \mathfrak{H} . Since ρ is a $*$ -representation, the matrix $(\rho(A_{ij}))$ is a non-negative operator on $\mathfrak{K} \oplus \dots \oplus \mathfrak{K}$; and therefore,

$$\sum_{i,j} (\mu(A_{ij}) x_j, x_i) = \sum_{i,j} (\rho(A_{ij}) V x_j, V x_i) \geq 0.$$

PROOF OF SUFFICIENCY. Suppose that μ is completely positive. Consider the vector space $\mathcal{A} \otimes \mathfrak{H}$, the algebraic tensor product of \mathcal{A} and \mathfrak{H} . For $\xi = \sum_i A_i \otimes x_i$ and $\eta = \sum_j B_j \otimes y_j$ in $\mathcal{A} \otimes \mathfrak{H}$ we define

$$(\xi, \eta) = \sum_{i,j} (\mu(B_j^* A_i) x_i, y_j).$$

Since μ was assumed to be completely positive, it follows that

$$(\xi, \xi) = \sum_{i,j} (\mu(A_j^* A_i) x_i, x_j) \geq 0.$$

Hence (\cdot, \cdot) is a positive Hermitian bilinear form. There is a natural mapping ρ' from \mathcal{A} to linear transformations on $\mathcal{A} \otimes \mathfrak{H}$ given by

$$\rho'(A) \sum_i B_i \otimes y_i = \sum_i (A B_i) \otimes y_i.$$

We shall show that for all A in \mathcal{A} and ξ in $\mathcal{A} \otimes \mathfrak{H}$

$$(1) \quad (\rho'(A)\xi, \rho'(A)\xi) \leq \|A\|^2 (\xi, \xi).$$

If (1) were not universally true, we could find A in \mathcal{A} and

$$\xi = \sum_i B_i \otimes x_i$$

in $\mathcal{A} \otimes \mathfrak{H}$ such that $(\xi, \xi) \leq 1$ and $\|A\| < 1$, but $(\rho'(A)\xi, \rho'(A)\xi) > 1$.

Then $(\rho'(A^*A)\xi, \xi) > 1$, which implies, by the Schwarz inequality, that $(\rho'(A^*A)\xi, \rho'(A^*A)\xi) > 1$. By continuing in this manner, we find that

$$(\rho'([A^*A]^{2^k})\xi, \xi) > 1 \quad \text{for } k = 1, 2, \dots$$

Since μ is positive and $-\|C\|1 \leq C \leq \|C\|1$ whenever C is a self-adjoint element of \mathcal{A} , it follows that $-\|C\|\mu(1) \leq \mu(C) \leq \|C\|\mu(1)$ and hence that $\|\mu(C)\| \leq \|C\|\|\mu(1)\|$; this inequality shows that μ is uniformly continuous on the self-adjoint elements of \mathcal{A} , and it is easy to see from this that μ must be uniformly continuous on all of \mathcal{A} . The uniform continuity of μ together with the fact that $\|A\| < 1$ shows that

$$(\rho'([A^*A]^{2^k})\xi, \xi) = \sum_{i,j} (\mu(B_j^*[A^*A]^{2^k}B_i)x_i, x_j)$$

converges to 0. This contradiction proves (1).

Let \mathcal{N} be the set of all ξ in $\mathcal{A} \otimes \mathfrak{K}$ such that $(\xi, \xi) = 0$. By the Schwarz inequality, \mathcal{N} is a linear manifold and by (1), \mathcal{N} is invariant under $\rho'(\mathcal{A})$. Therefore, the quotient space $\mathcal{A} \otimes \mathfrak{K} / \mathcal{N}$ is a pre-Hilbert space, and each A in \mathcal{A} naturally induces a bounded operator on the completion \mathcal{K} of $\mathcal{A} \otimes \mathfrak{K} / \mathcal{N}$. Let

$$Vx = 1 \otimes x + \mathcal{N} \quad \text{for all } x \in \mathfrak{K}.$$

Then $\|Vx\|^2 \leq (\mu(1)x, x)$, so that V is a bounded linear transformation from \mathfrak{K} to \mathcal{K} . Now

$$\begin{aligned} (V^*\rho(A)Vx, y) &= (\rho(A)Vx, Vy) = (\rho'(A)1 \otimes x, 1 \otimes y) \\ &= (A \otimes x, 1 \otimes y) = (\mu(A)x, y) \end{aligned}$$

for all x and y in \mathfrak{K} , and therefore $\mu(A) = V^*\rho(A)V$.

3. Remarks. The operator $\rho(1)$ is a projection. We can take $\rho(1)$ to be 1, for we can replace V by $\rho(1)V$ and then replace the space \mathcal{K} by the subspace $\rho(1)\mathcal{K}$. Assuming that this has been done, we have $\mu(1) = V^*V$, so that if $\mu(1) = 1$, then V is an isometry. Since an isometry can be considered as an embedding, the Neumark theorem follows from Theorem 1, provided we can show that when \mathcal{A} is commutative, positivity of μ implies complete positivity. This fact will be proved in the next section.

It might be thought possibly that positivity always implies complete positivity. We give a counter-example to show that this is not the case. Let \mathcal{A} be the algebra of $n \times n$ matrices with complex entries. We denote by e_{ij} the matrix with 1 in the i th row and j th column and

with zeros elsewhere. The e_{ij} 's are a basis for the vector space \mathcal{A} . We define a linear function μ from \mathcal{A} to \mathcal{A} by specifying the values of μ on this basis: $\mu(e_{ij})$ is to be a matrix whose (r, s) th entry is $(r-j)(s-i)$. Now μ is positive; for let $A = (a_{ij}) \geq 0$ be an $n \times n$ matrix and let

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

be any vector. Then

$$\begin{aligned} (\mu(A)x, x) &= \sum_{i,j,r,s} a_{ij}(r-j)(s-i)x_r\bar{x}_s \\ &= \sum_{i,j} a_{ij} \left(\sum_r (r-j)x_r \right) \left(\sum_s (s-i)\bar{x}_s \right) \geq 0 \end{aligned}$$

since $A \geq 0$.

On the other hand, μ is not completely positive, for we shall show that $\mu^{(n)}$ is not positive, n being as above. Let E be the element of $\mathcal{A}^{(n)}$ whose (i, j) th entry is the matrix e_{ij} . It is easily seen that $E \geq 0$. But $\mu^{(n)}(E)$ is not ≥ 0 , for

$$\sum_{i,j,r,s} (r-j)(s-i)\delta_{ir}\delta_{js} = \sum_{i,j} (i-j)(j-i) < 0.$$

4. Conditions for complete positivity.

THEOREM 2. *If \mathcal{A} is a W^* -algebra of finite type (see [1]), then the center-valued trace t is completely positive.*

PROOF. Suppose $M = (A_{ij})$ is a non-negative matrix in $\mathcal{A}^{(n)}$ and suppose $\epsilon > 0$. It is shown in [1] that there exist a finite number U_1, \dots, U_r of unitary operators in \mathcal{A} and non-negative numbers $\alpha_1, \dots, \alpha_r$ with $\sum_k \alpha_k = 1$ such that

$$\left\| t(A_{ij}) - \sum_k \alpha_k U_k^* A_{ij} U_k \right\| < \epsilon \quad \text{for } i, j = 1, \dots, n.$$

But then if x_1, \dots, x_n are any vectors,

$$\left| \sum_{i,j} (t(A_{ij})x_j, x_i) - \sum_{i,j,k} \alpha_k (U_k^* A_{ij} U_k x_j, x_i) \right| \leq \epsilon \sum_{i,j} |(x_j, x_i)|$$

and it is clear that

$$\sum_{i,j,k} \alpha_k (U_k^* A_{ij} U_k x_j, x_i) = \sum_k \alpha_k \sum_{i,j} (A_{ij} U_k x_j, U_k x_i) \geq 0$$

since $M \geq 0$. It follows that $t^{(n)}(M) \geq 0$.

THEOREM 3. *If \mathcal{A} is a C^* -algebra and μ is a positive linear function with complex values, then μ is completely positive.*

PROOF. Suppose $M = (A_{ij})$ is a non-negative matrix in $\mathcal{A}^{(n)}$, and $\{\lambda_i\}$ are complex numbers. We wish to check that

$$\sum_{i,j} \mu(A_{ij}) \lambda_j \bar{\lambda}_i \geq 0.$$

But $\sum_{i,j} \mu(A_{ij}) \lambda_j \bar{\lambda}_i = \mu(\sum_{i,j} A_{ij} \lambda_j \bar{\lambda}_i)$. Writing $M = N^*N$ with $N = (B_{ij})$, we see that

$$\begin{aligned} \sum_{i,j} A_{ij} \lambda_j \bar{\lambda}_i &= \sum_{i,j} \lambda_j \bar{\lambda}_i \sum_k B_{ki}^* B_{kj} \\ &= \sum_k \left(\sum_i \lambda_i B_{ki} \right)^* \left(\sum_j \lambda_j B_{kj} \right) \geq 0 \end{aligned}$$

and so $\mu(\sum_{i,j} A_{ij} \lambda_j \bar{\lambda}_i) \geq 0$ since μ is positive.

Theorem 3 together with Theorem 1 gives the known fact that a state of a C^* -algebra induces a representation [3].

THEOREM 4. *If \mathcal{A} is a commutative C^* -algebra and μ is a positive operator-valued linear function on \mathcal{A} , then μ is completely positive.*

PROOF. We may take \mathcal{A} as the algebra of all continuous complex-valued functions vanishing at ∞ on a locally compact Hausdorff space Γ . Let $M = (f_{ij})$ be a non-negative matrix in $\mathcal{A}^{(n)}$. If x_1, \dots, x_n are vectors in the Hilbert space, we wish to verify that

$$\sum_{i,j} (\mu(f_{ij}) x_j, x_i) \geq 0.$$

By the Riesz-Markoff theorem, there exists a regular measure m on Γ such that $\sum_i (\mu(f) x_i, x_i) = \int_{\Gamma} f dm$ for all $f \in \mathcal{A}$. Then by the Riesz-Markoff and Radon-Nikodym theorems, there exist measurable functions h_{ij} such that

$$(\mu(f) x_j, x_i) = \int_{\Gamma} f h_{ij} dm \quad \text{for all } f \in \mathcal{A}.$$

Now the matrix $(h_{ij}(\gamma))$ is non-negative almost everywhere; for

$$\int_{\Gamma} f \sum_{i,j} h_{ij} \lambda_i \bar{\lambda}_j dm \geq 0 \quad \text{for all } f \geq 0 \text{ in } \mathcal{A},$$

and hence $\sum_{i,j} h_{ij}(\gamma) \lambda_i \bar{\lambda}_j \geq 0$ for all r -tuples $\lambda_1, \dots, \lambda_r$ of complex numbers with rational real and imaginary parts and for all $\gamma \in \Gamma$ with the exception of $\gamma \in N$ where $m(N) = 0$. Also the matrix $(f_{ij}(\gamma))$ is

non-negative for all $\gamma \in \Gamma$ by Theorem 3. Therefore,

$$\sum_{i,j} f_{ij}(\gamma) h_{ij}(\gamma) \geq 0 \quad \text{almost everywhere}$$

and so

$$\sum_{i,j} (\mu(f_{ij}) x_j, x_i) = \int_{\Gamma} \sum_{i,j} f_{ij} h_{ij} dm \geq 0.$$

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