AN ISOMORPHISM THEOREM FOR FINITELY ADDITIVE MEASURES

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A problem which is appealing to the intuition in view of the relative frequency interpretation of probability is to define a measure on a countable space which assigns to each point the measure 0. Such a measure of course becomes trivial if it is countably additive. Finitely additive measures of this type have been discussed by R. C. Buck [1] and by E. F. Buck and R. C. Buck [2]. In a discussion of the density of sets of integers, R. C. Buck introduces in [1] a special finitely additive measurable space, $\mathcal{D}_0^*$, containing the arithmetic progressions and assigning to each a measure $m^*$ equal to the reciprocal of their period. Special properties of this measure are developed there largely from the number theoretic point of view. In [2] the authors showed that any separable, non atomic, normalized, finitely additive measure was isomorphic to a contraction of $[\mathcal{D}_0^*, m^*]$.

Necessary and sufficient conditions for a separable, non atomic measure to be point and set isomorphic to the Borel sets have been established by Halmos and von Neumann [3]. Here we consider the analogous problem for a finitely additive measure on a countable space. We show that a wide class of finitely additive measures in a countable space are set isomorphic to Jordan content on [0, 1] and point isomorphic to a restriction of Jordan content to a countable dense subset of [0, 1].

Let $X$ be a countable space and let $\mu_P$ be a real, finitely additive function defined on a countable class $\mathcal{P}$ of subsets of $X$. We assume the following properties of $\mathcal{P}$ and $\mu_P$:

(i) $E_1, E_2 \in \mathcal{P}$ implies $E_1 \cap E_2 \in \mathcal{P}$,
(ii) $E_1, E_2 \in \mathcal{P}, E_1 \subseteq E_2$ implies there exists $C_i$ such that $E_1 = C_1 \subseteq C_2 \subseteq \cdots \subseteq C_n = E_2$ and $C_i - C_{i-1} \in \mathcal{P}$,
(iii) $X$ and 0 are in $\mathcal{P}$ and $\mu_P(X) = 1$, $\mu_P(0) = 0$, further $E_1 \in \mathcal{P}$, $E_1 \neq 0$ implies that $\mu_P(E_1) > 0$.
(iv) For any $s \in X$ there exist $E_i \in \mathcal{P}$ such that $s \in E_i$ and
$$\lim_{i} \mu_P E_i = 0.$$ The sets $\mathcal{P}$ then form a semi-ring (cf. [4, p. 22]), and there is a unique minimal ring $\mathcal{R}(\mathcal{P})$ containing $\mathcal{P}$ with a finitely additive measure $\mu_R$ such that $\mu_R E = \mu_P E$ if $E \in \mathcal{P}$. Indeed it is easy to show

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that $\mathcal{R}(\mathcal{P})$ consists of just those sets $E$ which are finite, disjoint unions of sets $E_1 \cdots E_n$ in $\mathcal{P}$ and $\mu_R E = \sum \mu_R E_i$. We consider measures $[\mathcal{M}, \mu]$ which are the finite Carathéodory closures of such a collection $[\mathcal{R}(\mathcal{P}), \mu_R]$ (i.e. $E$ is in $\mathcal{M}$ if and only if, for any $\epsilon > 0$ there are sets $G$ and $H$ in $\mathcal{R}(\mathcal{P})$ such that $G \supset E$, $H \supset X - E$ and $\mu_R (G \cap H) < \epsilon$; in which case we define $\mu E = \text{glb} \{ \mu_R G; G \in \mathcal{R}(\mathcal{P}), G \supset E \}$). We shall refer to the measure spaces $[\mathcal{M}, \mu]$ as $f$-a measures. It may be remarked that the class $\mathcal{D}_0^*$ may be defined this way by taking the arithmetic progressions as the semi-ring $\mathcal{P}$. As a second example it may be noted that a finitely additive measure can be defined on a countable, dense set $D$ on the interval $[0, 1)$ by defining a set $E$ to be measurable if its closure $\text{Cl} E$ is in the class $\mathcal{J}$ of sets possessing Jordan content and letting the measure of the set, $\nu E$, be the content $m \text{Cl} E$ of its closure. Such a measure arises from the above construction if we let $\mathcal{P}$ be the collection of all sets which are the intersection of $D$ with half open rational sub-intervals of $[0, 1)$. We show that such a measure space $[\mathcal{C}, \nu]$ is universal in the sense that if $[\mathcal{M}, \mu]$ is any $f$-a measure, then there is a 1-1 transformation $T$ such that $T(X)$ is a dense set $D$ on $[0, 1)$ and $\mu E = \nu T(E)$ when $E$ is in $\mathcal{M}$.

**Theorem 1.** For any $f$-a measure $[\mathcal{M}, \mu]$ there exists a 1-1 transformation $T$ carrying $X$ into a dense set $D$ on $[0, 1)$ and satisfying the above condition. Furthermore for any $x \in X$ the numerical value of $T(x)$ can be determined by a finite number of arithmetic operations.

Let $\{x_i\}$, $\{E_i\}$ be orderings of the points of $X$ and the sets of $\mathcal{P}$. We shall describe by induction a sub-ordering $E_i^{(j)}$ of $\mathcal{P}$ (possibly containing some sets more than once) such that for each $i$, $E_i^{(j)}$ is a finite union of sets exhausting $X$ and containing a sub-union equal to $E_i$ and such that each set $E_i^{(j)}$ is a subset of some set $E_l^{(j)}$. Let $E_0 = X$ and suppose sets $E_i^{(j)}$, $i=0, 1, 2, \cdots, k-1$, $j=1, 2, \cdots, n_i$, have been described; we now determine the sets $E_i^{(j)}$, $j=1, 2, \cdots, n_k$. Let $E_k^{(1)} = E_k \cap E_{k-1}^{(1)}$ if this set is not null; in which case by (ii) above there exist sets $E_k^{(j)}$ ($j=2, \cdots, n_1$) such that $E_k^{(i)} \cap E_k^{(j)} = 0$ $(i \neq j)$, $E_k^{(j)} \subset E_{k-1}^{(1)}$, $\bigcup_{n_1} E_k^{(j)} = E_k^{(1)}$, and $E_k^{(1)} \in \mathcal{P}$. If $E_k \cap E_{k-1}^{(1)} = 0$ we let $E_k^{(1)} = E_{k-1}^{(1)}$. We define a partition of each of the sets $E_k^{(j)}$ similarly and the resulting sets $E_k^{(j)}$ have the desired properties. By finite additivity of the function $\mu$ we see that if $E_k^{(i)}$ ($i=1, 2, \cdots, n_j$) are the subsets of $E_k^{(j)}$ contained in $E_k^{(j)}$, then $\sum_{n_1} \mu E_k^{(j)} = \mu E_k^{(j)}$. We shall denote by $x_i^{(j)}$ the point $x_{i'}$ where $i'$ is the minimum $n$ such that $x_n$ is in $E_i^{(j)}$. Set transformations $T_k$ can be defined on $E_k^{(j)}$ such that $T_k(E_k^{(j)})$ is an interval (closed on the left, open on the right) on $[0, 1)$ for $k \geq i$ and whose length preserves the finitely additive $\mu$.
measure on $E_k^{(0)}$ and such that if $x_i^k = x_j^k$ then $T_k(E_i^{(0)})$ and $T_k(E_j^{(0)})$, $k \geq \max (i_1, i_2)$, have the same left-hand end points. Let $T_0 \{E_0^{(0)}\} = [0, 1)$ and suppose $T_i$ have been defined having the above properties for $i = 0, 1, \ldots, k - 1$. Let $E_i^{(0)} (i = 0, 1, 2, \ldots, n^i)$ be the sets of $E_k^{(0)}$ contained in $E_i^{(0)}$ ordered such that $E_i^{(0)} = \emptyset$ and $x_i^{(0)} = x_{k-1}^{(0)}$. We define $T_k$ on these sets such that $T_k(E_i^{(0)}) = [a + \sum_{j=1}^{n^i} \mu E_j^{(0)}], a + \sum_{j=1}^{n^i} \mu E_j^{(0)}$ where $a$ is the left-hand end point of $T_{k-1}[E_{k-1}^{(0)}]$; (by property (iii)) $E_i^{(0)} \neq \emptyset$ implies $\mu E_i^{(0)} > 0$ and hence $T_k(E_i^{(0)})$ is a nondegenerate interval. $T_k$ is defined similarly on the remaining sets of $E_k^{(0)}$ and we see that the collection of transformations $T_k (k = 0, 1, 2 \ldots)$ has the desired properties.

We denote by $D$ the set of left-hand end points of intervals of the form $T_k[E_k^{(0)}]$; $D$ is then dense on $[0, 1)$ by property (iv). But by the construction of $T_k$ and property (iv) the point $x_i^{(0)}$ is the unique point contained in all intervals $T_i(E_i^{(0)})$ having the same left-hand end point as $T_k(E_i^{(0)})$. Thus we may define a 1-1 transformation $T$ such that $T[x_i^{(0)}]$ is the left-hand end point of $T_k[E_k^{(0)}]$, and then $T(X) = D$. Let $E_k^{(0)}$ be any set in $\mathcal{P}$ and let $x \in E_k^{(0)}$; then $T(x) \in T_k(E_k^{(0)})$. This implies (a) $\mu \text{Cl} T(E_k^{(0)}) \leq \mu T_k(E_k^{(0)}) = \mu E_k^{(0)}$; whereas by the density of $D$ we have that (b) $\sum_{j=1}^{n^i} \mu \text{Cl} T[E_j^{(0)}] \geq 1$. Applying the finite additivity of $\mu$ on $\mathcal{P}$ we see that (c) $\sum_{j=1}^{n^i} \mu E_j^{(0)} = 1$, and combining a, b, and c we see that $\mu \text{Cl} T[E_k^{(0)}] = \mu E_k^{(0)}$. This completes the proof since clearly $T[R(\mathcal{P})] = \mathcal{R}[T[\mathcal{P}]]$ and $[C, \nu]$ is the finite Carathéodory closure of $T(\mathcal{P})$ and hence also of $T[R(\mathcal{P})]$.

We may now describe the set isomorphism $\tau$ which arises rather naturally between the classes $[\mathcal{J}, \mu]$ and $[\mathcal{C}, \nu]$. Let $E \Delta F$ denote the symmetric difference $(E - F) \cup (F - E)$. If $\mathcal{E}$ is a subclass of $\mathcal{C}$ such that $E, F \in \mathcal{E}$ implies $\nu(E \Delta F) = 0$ then $\tau(\mathcal{E})$ is defined as that sub-class $\mathcal{F}^* \subseteq \mathcal{J}$ which contains ClE for some $E \in \mathcal{E}$ and satisfies $E, F \in \mathcal{F}^*$ implies $\mu(E \Delta F) = 0$. We note several differences between this isomorphism and the point isomorphism described in the theorem. First the correspondence is not 1-1 but modulo sets of measure 0. More important, however, if $E_i$ is a countable, disjoint collection of sets in $\mathcal{C}$ then in order that $\bigcup E_i$ be in $\mathcal{C}$ it is sufficient but not necessary that $\bigcup \text{Cl}E_i$ be in $\mathcal{J}$.

Some properties of general f-a measures follow immediately from the corresponding properties for Jordan content. Thus we see that if $E$ is in $\mathcal{M}$ and $\mu E = \alpha > 0$ then the range of the measure of subsets of $E$ contained in $\mathcal{M}$ is the interval $[0, \alpha]$. Any set $F$ possessing Jordan content differs from an open set by a set of measure 0 (i.e. $\mu(\text{Cl}F$...
\( \cap \text{Cl}(F') = 0 \). Hence, using the density of the partitions \( \{E^{(i)}_t\} \), for any \( E \in \mathcal{M} \) there are sets \( P_t \in \mathcal{P} \) such that, \( \cup P_t \in \mathcal{M} \) and \( \mu(\cup P_t \Delta E) = 0 \). In particular for the class \( \mathcal{D}^*_0 \) we see that any measurable set differs from a union of arithmetic progressions by a set of measure 0.

If \( E_j \) is a disjoint sequence of sets on \([0, 1]\), if \( S_1 = \bigcup [\text{Cl}E_j \cap \text{Cl}E'_j] \), \( S_2 = \{x; x \in \text{Cl}E_j, x \in \text{Cl} \cup E_j\} \) and \( S_3 = \text{Cl} \cup E_j \cap \text{Cl}(\cup E_j)' \) then \( mS_1 = 0 \) if and only if \( E_j \in \mathcal{J} \) for all \( j \), \( mS_2 = 0 \) if and only if \( m \text{Cl} \cup E_j = \sum m \text{Cl} E_j \) and \( mS_3 = 0 \) if and only if \( \cup E_j \in \mathcal{J} \). Since \( S_1 \cup S_2 \supset S_3 \) and content is numerically equal to Lebesgue measure we see that if \( E_j \in \mathcal{J} \) then a necessary and sufficient condition that \( \cup E_j \in \mathcal{J} \) is that \( m \text{Cl} \cup E_j = \sum m \text{Cl} E_j \). If this condition holds and if \( K_j \in \mathcal{J} \), \( K_j \subset E_j \), then \( \cup K_j \in \mathcal{J} \) and \( m \cup K_j = \sum mK_j \). To see this, let \( S_4 = \text{Cl}(\text{Cl}K_j \cap \text{Cl}K'_j) \) then \( mS_4 = 0 \); then \( \text{Cl} \cup K_j \cap \text{Cl}(\cup K_j)' = K \subset S_1 \cup S_2 \cup S_4 \) for if \( x \in K \) and \( x \in E_i \) for some \( i \), then \( x \in K_i \) implies \( x \in S_4 \), \( x \notin K_i \) implies \( x \in S_1 \), whereas \( x \notin E_i \) for any \( i \) implies \( x \in S_2 \). By Theorem 1 these properties of \( \mathcal{J} \) hold for general \( f \)-a measures namely:

**Theorem 2.** If \( E_j \) is a sequence of disjoint sets in \( \mathcal{M} \), then a necessary and sufficient condition that \( \cup E_j \in \mathcal{J} \) is in \( \mathcal{M} \) with \( \mu(\cup E_j) = \sum \mu E_j \) is that \( \cup \text{Cl}(E_j) \) be in \( \mathcal{J} \). If this condition holds and if \( K_j \) are in \( \mathcal{M} \) with \( K_j \subset E_j \) then \( \cup K_j \) is in \( \mathcal{M} \) and \( \mu(\cup K_j) = \sum \mu K_j \).

The last statement of the theorem may be applied to show, for example, that in \( \mathcal{D}^*_0 \) the set of all primes is measurable and has measure 0, a fact noted by Buck [1]. This follows since we may find a sequence of arithmetic progressions, each containing one prime whose union is the space of all positive integers and with sum of measures 1.

**Bibliography**


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