AN ISOMORPHISM THEOREM FOR FINITELY ADDITIVE MEASURES

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A problem which is appealing to the intuition in view of the relative frequency interpretation of probability is to define a measure on a countable space which assigns to each point the measure 0. Such a measure of course becomes trivial if it is countably additive. Finitely additive measures of this type have been discussed by R. C. Buck [1] and by E. F. Buck and R. C. Buck [2]. In a discussion of the density of sets of integers, R. C. Buck introduces in [1] a special finitely additive measurable space, $\mathcal{D}_0^*$, containing the arithmetic progressions and assigning to each a measure m^* equal to the reciprocal of their period. Special properties of this measure are developed there largely from the number theoretic point of view. In [2] the authors showed that any separable, non atomic, normalized, finitely additive measure was isomorphic to a contraction of $[\mathcal{D}_0^*, m^*]$.

Necessary and sufficient conditions for a separable, non atomic measure to be point and set isomorphic to the Borel sets have been established by Halmos and von Neumann [3]. Here we consider the analogous problem for a finitely additive measure on a countable space. We show that a wide class of finitely additive measures in a countable space are set isomorphic to Jordan content on $[0, 1]$ and point isomorphic to a restriction of Jordan content to a countable dense subset of $[0, 1]$.

Let X be a countable space and let μ_P be a real, finitely additive function defined on a countable class $\mathcal{P}$ of subsets of X. We assume the following properties of $\mathcal{P}$ and μ_P:

(i) $E_1, E_2 \in \mathcal{P}$ implies $E_1 \cap E_2 \in \mathcal{P}$,

(ii) $E_1, E_2 \in \mathcal{P}$, $E_1 \subset E_2$ implies there exists C_i such that $E_1 = C_1 \subset C_2 \subset \cdots \subset C_n = E_2$ and $C_i - C_{i-1} \in \mathcal{P}$,

(iii) X and 0 are in $\mathcal{P}$ and $\mu_P(X) = 1$, $\mu_P(0) = 0$, further $E_1 \in \mathcal{P}$, $E_1 \neq 0$ implies that $\mu_P(E_1) > 0$.

(iv) For any $s \in X$ there exist $E_i \in \mathcal{P}$ such that $s \in E_i$ and

$$\lim_i \mu_P E_i = 0.$$

The sets $\mathcal{P}$ then form a semi-ring (cf. [4, p. 22]), and there is a unique minimal ring $\mathcal{R}(\mathcal{P})$ containing $\mathcal{P}$ with a finitely additive measure μ_R such that $\mu_R E = \mu_P E$ if $E \in \mathcal{P}$. Indeed it is easy to show

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that $\mathcal{R}(\mathcal{P})$ consists of just those sets E which are finite, disjoint unions of sets $E_1 \cdots E_n$ in $\mathcal{P}$ and $\mu_R E = \sum \mu_P E_i$. We consider measures $[\mathcal{M}, \mu]$ which are the finite Carathéodory closures of such a collection $[\mathcal{R}(\mathcal{P}), \mu_R]$ (i.e. E is in $\mathcal{M}$ if and only if, for any $\epsilon > 0$ there are sets G and H in $\mathcal{R}(\mathcal{P})$ such that $G \supset E$, $H \supset X - E$ and $\mu_R(G \cap H) < \epsilon$; in which case we define $\mu E = \text{glb } [\mu_R \mathcal{G};\ \mathcal{G} \in \mathcal{R}(\mathcal{P}),\ \mathcal{G} \supset E]$). We shall refer to the measure spaces $[\mathcal{M}, \mu]$ as f-a measures. It may be remarked that the class $\mathcal{D}_0^*$ may be defined this way by taking the arithmetic progressions as the semi-ring $\mathcal{P}$. As a second example it may be noted that a finitely additive measure can be defined on a countable, dense set D on the interval $[0, 1)$ by defining a set E to be measurable if its closure $\text{Cl}E$ is in the class $\mathcal{J}$ of sets possessing Jordan content and letting the measure of the set, νE, be the content $m\text{Cl}E$ of its closure. Such a measure arises from the above construction if we let $\mathcal{P}$ be the collection of all sets which are the intersection of D with half open rational sub-intervals of $[0, 1)$. We show that such a measure space $[\mathcal{C}, \nu]$ is universal in the sense that if $[\mathcal{M}, \mu]$ is any f-a measure, then there is a 1-1 transformation T such that $T(X)$ is a dense set D on $[0, 1)$ and $\mu E = \nu T(E)$ when E is in $\mathcal{M}$.

THEOREM 1. *For any f-a measure $[\mathcal{M}, \mu]$ there exists a* 1-1 *transformation T carrying X into a dense set D on $[0, 1)$ and satisfying the above condition. Furthermore for any $x \in X$ the numerical value of $T(x)$ can be determined by a finite number of arithmetic operations.*

Let $\{x_i\}$, $\{E_i\}$ be orderings of the points of X and the sets of $\mathcal{P}$. We shall describe by induction a sub-ordering $E_i^{(j)}$ of $\mathcal{P}$ (possibly containing some sets more than once) such that for each i, $E_i^{(j)}$ is a finite union of sets exhausting X and containing a sub-union equal to E_i and such that each set $E_i^{(j)}$ is a subset of some set $E_{i-1}^{(k)}$. Let $E_0^1 = X$ and suppose sets $E_i^{(j)}$, $i = 0, 1, 2, \cdots, k-1$, $j = 1, 2, \cdots, n_i$, have been described; we now determine the sets $E_k^{(j)}$, $j = 1, 2, \cdots, n_k$. Let $E_k^{(1)} = E_k \cap E_{k-1}^{(1)}$ if this set is not null; in which case by (ii) above there exist sets $E_k^{(j)}$ $(j = 2, \cdots, n_1)$ such that $E_k^{(i)} \cap E_k^{(j)} = 0$ $(i \neq j)$, $E_k^{(j)} \subset E_{k-1}^{(1)}$, $\bigcup_{j=1}^{n_1} E_k^{(j)} = E_{k-1}^{(1)}$, and $E_k^{(j)} \in \mathcal{P}$. If $E_k \cap E_{k-1}^{(1)} = 0$ we let $E_k^{(1)} = E_{k-1}^{(1)}$. We define a partition of each of the sets $E_{k-1}^{(j)}$ similarly and the resulting sets $E_k^{(j)}$ have the desired properties. By finite additivity of the function μ we see that if $E_k^{(j_i)}$ $(i = 1, 2, \cdots, n_j)$ are the subsets of $E_k^{(i)}$ contained in $E_{k-1}^{(j)}$, then $\sum_{i=1}^{n_j} \mu E_k^{j_i} = \mu E_{k-1}^{(j)}$. We shall denote by $x_i^{(j)}$ the point $x_{i'}$ where i' is the minimum n such that x_n is in $E_i^{(j)}$. Set transformations T_k can be defined on $E_k^{(j)}$ such that $T_k(E_i^{(j)})$ is an interval (closed on the left, open on the right) on $[0, 1)$ for $k \geqq i$ and whose length preserves the finitely additive μ

measure on $E_k^{(j)}$ and such that if $x_{i_1}^{j_1}=x_{i_2}^{j_2}$ then $T_k(E_{i_1}^{(j_1)})$ and $T_k(E_{i_2}^{(j_2)})$, $k\geqq\max(i_1, i_2)$, have the same left-hand end points. Let $T_0[E_0^{(1)}] = [0, 1)$ and suppose T_i have been defined having the above properties for $i=0, 1, \cdots, k-1$. Let $E_k^{(j_i)}$ $(i=0, 1, 2, \cdots, n_j')$ be the sets of $E_k^{(i)}$ contained in $E_{k-1}^{(j)}$ ordered such that $E_k^{(j_0)}=0$ and $x_k^{(j_1)}=x_{k-1}^{(j)}$. We define T_k on these sets such that $T_k(E_k^{(j_i)})=[a+\sum_{t=1}^{j_i-1}\mu E_k^{(j_t)}, a+\sum_{t=1}^{j_i}\mu E_k^{(j_t)})$ where a is the left-hand end point of $T_{k-1}[E_{k-1}^{(j)}]$; (by property (iii)) $E_k^{(t)}\neq 0$ implies $\mu E_k^{(t)}>0$ and hence $T_k(E_k^{(j_i)})$ is a nondegenerate interval. T_k is defined similarly on the remaining sets of $E_k^{(j)}$ and we see that the collection of transformations T_k $(k=0, 1, 2 \cdots)$ has the desired properties.

We denote by D the set of left-hand end points of intervals of the form $T_k[E_k^{(j)}]$; D is then dense on $[0, 1)$ by property (iv). But by the construction of T_k and property (iv) the point $x_k^{(j)}$ is the unique point contained in all intervals $T_i(E_i^{(j)})$ having the same left-hand end point as $T_k(E_k^{(j)})$. Thus we may define a 1-1 transformation T such that $T[x_k^{(j)}]$ is the left-hand end point of $T_k[E_k^{(j)}]$, and then $T(X)=D$. Let $E_k^{(j)}$ be any set in $\mathcal{P}$ and let $\bar{x}\in E_k^{(j)}$; then $T(\bar{x})\in T_k(E_k^{(j)})$. This implies (a) $m\mathrm{Cl}T(E_k^{(j)})\leqq mT_k(E_k^{(j)})=\mu E_k^{(j)}$; whereas by the density of D we have that (b) $\sum_{j=1}^{n_k} m\mathrm{Cl}T[E_k^{(j)}]\geqq 1$. Applying the finite additivity of μ on $\mathcal{P}$ we see that (c) $\sum_{j=1}^{n_k}\mu E_k^{(j)}=1$, and combining a, b, and c we see that $m\mathrm{Cl}T[E_k^{(j)}]=\mu E_k^{(j)}$. This completes the proof since clearly $T[\mathcal{R}(\mathcal{P})]=\mathcal{R}T[\mathcal{P}]$ and $[C, \nu]$ is the finite Carathéodory closure of $T(\mathcal{P})$ and hence also of $T[\mathcal{R}(\mathcal{P})]$.

We may now describe the set isomorphism τ which arises rather naturally between the classes $[\mathcal{J}, m]$ and $[\mathcal{C}, \nu]$. Let $E\Delta F$ denote the symmetric difference $(E-F)\cup(F-E)$. If $\mathcal{E}$ is a subclass of $\mathcal{C}$ such that $E, F\in\mathcal{E}$ implies $\nu(E\Delta F)=0$ then $\tau(\mathcal{E})$ is defined as that sub-class $\mathcal{J}^*$ of $\mathcal{J}$ which contains $\mathrm{Cl}E$ for some $E\in\mathcal{E}$ and satisfies $E, F\in\mathcal{J}^*$ implies $m(E\Delta F)=0$. We note several differences between this isomorphism and the point isomorphism described in the theorem. First the correspondence is not 1-1 but modulo sets of measure 0. More important, however, if E_i is a countable, disjoint collection of sets in $\mathcal{C}$ then in order that $\bigcup E_i$ be in $\mathcal{C}$ it is sufficient but not necessary that $\bigcup\mathrm{Cl}E_i$ be in $\mathcal{J}$.

Some properties of general f-a measures follow immediately from the corresponding properties for Jordan content. Thus we see that if E is in $\mathcal{M}$ and $\mu E=\alpha>0$ then the range of the measure of subsets of E contained in $\mathcal{M}$ is the interval $[0, \alpha]$.[1] Any set F possessing Jordan content differs from an open set by a set of measure 0 (i.e. $m(\mathrm{Cl}F$

[1] It follows from a result of Sobczyk and Hammer that this set is perfect [5].

$\cap \mathrm{Cl} F')=0)$. Hence, using the density of the partitions $\{E_i^{(j)}\}$, for any $E \in \mathcal{M}$ there are sets $P_i \in \mathcal{P}$ such that, $\bigcup P_i \in \mathcal{M}$ and $\mu(\bigcup P_i \Delta E) = 0$. In particular for the class $\mathcal{D}_0^*$ we see that any measurable set differs from a union of arithmetic progressions by a set of measure 0.

If E_j is a disjoint sequence of sets on $[0, 1]$, if $S_1 = \bigcup [\mathrm{Cl} E_j \cap \mathrm{Cl} E_j']$, $S_2 = [x; x \notin \mathrm{Cl} E_j, x \in \mathrm{Cl} \bigcup E_j]$ and $S_3 = \mathrm{Cl} \bigcup E_j \cap \mathrm{Cl}(\bigcup E_j)'$ then $mS_1 = 0$ if and only if $E_j \in \mathcal{J}$ for all j, $mS_2 = 0$ if and only if $m\mathrm{Cl} \bigcup E_j = \sum m \mathrm{Cl} E_j$ and $mS_3 = 0$ if and only if $\bigcup E_j \in \mathcal{J}$. Since $S_1 \cup S_2 \supset S_3$ and content is numerically equal to Lebesgue measure we see that if $E_j \in \mathcal{J}$ then a necessary and sufficient condition that $\bigcup E_j \in \mathcal{J}$ is that $m\mathrm{Cl} \bigcup E_j = \sum m \mathrm{Cl} E_j$. If this condition holds and if $K_j \in \mathcal{J}$, $K_j \subset E_j$, then $\bigcup K_j \in \mathcal{J}$ and $m \cup K_j = \sum m K_j$. To see this, let $S_4 = \bigcup (\mathrm{Cl} K_j \cap \mathrm{Cl} K_j')$ then $mS_4 = 0$; then $\mathrm{Cl} \bigcup K_i \cap \mathrm{Cl}(\bigcup K_i)' = K \subset S_1 \cup S_2 \cup S_4$ for if $x \in K$ and $x \in E_i$ for some i, then $x \in K_i$ implies $x \in S_4$, $x \notin K_i$ implies $x \in S_1$, whereas $x \notin E_i$ for any i implies $x \in S_2$. By Theorem 1 these properties of $\mathcal{J}$ hold for general f-a measures namely:

THEOREM 2. *If E_j is a sequence of disjoint sets in $\mathcal{M}$, then a necessary and sufficient condition that $\bigcup E_j$ is in $\mathcal{M}$ with $\mu(\bigcup E_j) = \sum \mu E_j$ is that $\bigcup \mathrm{Cl} T(E_j)$ be in $\mathcal{J}$. If this condition holds and if K_j are in $\mathcal{M}$ with $K_j \subset E_j$ then $\bigcup K_j$ is in $\mathcal{M}$ and $\mu(\bigcup K_j) = \sum \mu K_j$.*

The last statement of the theorem may be applied to show, for example, that in $\mathcal{D}_0^*$ the set of all primes is measurable and has measure 0, a fact noted by Buck [1]. This follows since we may find a sequence of arithmetic progressions, each containing one prime whose union is the space of all positive integers and with sum of measures 1.

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