INTEGRALS FOR ASYMPTOTIC EXPANSIONS OF
HYPERGEOMETRIC FUNCTIONS

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1. The integral for ordinary hypergeometric functions. In this paper I discuss integrals which provide explicit asymptotic expansions of generalized basic hypergeometric functions. The problem of asymptotic expansions for hypergeometric functions has been considered previously by C. S. Maijer [2], E. M. Wright [5], and, for basic functions, by G. N. Watson [4].

Let

\[ \sum_{n=0}^{\infty} \frac{(a_1)_{n}(a_2)_{n} \cdots (a_A)_{n}z^n}{(b_1)_{n}(b_2)_{n} \cdots (b_B)_{n}n!} \]

where \((a)_n = a(a+1)(a+2) \cdots (a+n-1)\). Also let

\[ \frac{\Gamma(a_1)\Gamma(a_2) \cdots \Gamma(a_A)}{\Gamma(b_1)\Gamma(b_2) \cdots \Gamma(b_B)}. \]

A dash will denote the omission of a vanishing factor in a sequence. Thus, \((a)^' - a_r\) denotes the sequence \(a_1-a_r, a_2-a_r, \cdots, a_{r-1}-a_r, a_{r+1}-a_r, \cdots, a_A-a_r\).

It is known already (see [2] and [5]) that if

\[ I(z) = \frac{1}{2\pi i} \int_{-\infty}^{i\infty} \Gamma[(a) + s, (b) - s] z^s ds, \]

\[ \sum_A z^{-a} \Gamma[(a) - a, (b) + a] \]

\[ \times B+cF_{A+D-1} \left[ \begin{array}{c} (b) + a, 1 + a - (c); \\ 1 + a - (a), (d) + a; \end{array} \right] (-1)^{A+C}z^{-A}; \]

\[ \sum_B z^{b} \Gamma[(a) + b, (b) - b] \]

\[ \times A+dF_{B+C-1} \left[ \begin{array}{c} (a) + b, 1 + b - (d); \\ 1 + b - (b), (c) + b; \end{array} \right] (-1)^{B+D}z^{-B}; \]

then, provided that \(\pi |A+B-C-D|/2 > |\arg z|\),

(1.1) (i) \( I(z) = \sum_A z^{-a} \sim \sum_B z^{b} \) when \(B + C < A + D\),

(1.2) (ii) \( I(z) = \sum_B z^{b} \sim \sum_A z^{-a} \) when \(B + C > A + D\),

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and (iii) \( I(1) = \sum_A(1) = \sum_B(1), \) when \( A - C = B - D \geq 0 \) and

\[
(1.3) \quad R1 \sum(c + d - a - b) > 0.
\]

In particular, the cases \( A = 1, B = 2, C = D = 0, \) and \( A = B = C = 1, \)
\( D = 0, \) lead to

\[
\begin{align*}
1F1(a; b; z) & \sim \Gamma[1 + a - b; 1 - b]z^{-a}F_0[a, 1 + a - b; ; -1/z] \\
& \quad + e^{z^{a-b}}\Gamma[b; a]zF_0[1 - a, b - a; ; 1/z]
\end{align*}
\]

provided that \( |\arg z| < \pi/2. \) This is Barnes’ well-known result for
the confluent hypergeometric function (see \([1]\)).

2. The analogue for basic functions. I shall now state the corresponding results for basic hypergeometric functions together with
an outline of the proof. In the usual notation for basic series, let

\[
\Phi_B[(a); (b); z] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \cdots (a_A)_n}{(b_1)_n(b_2)_n \cdots (b_B)(a)_n} z^n
\]

where \((a)_n = (1 - q^a)(1 - q^{a+1}) \cdots (1 - q^{a+n-1}), \) and \(|q| < 1. \) Also let

\[
\prod P [(a); (b)] = \prod_{n=0}^{P} \frac{(1 - q^{a+n})(1 - q^{a+n+1}) \cdots (1 - q^{a+n+R})}{(1 - q^{b+n})(1 - q^{b+n+1}) \cdots (1 - q^{b+n+R})}.
\]

Let \( I_{P,R} \) be the integral

\[
\frac{1}{2\pi i} \int BC \prod [(a) + s, (b) - s, 1 - z + s, z - s; (c) + s, (d) - s]ds
\]

\[
= \int AD \prod (s)ds
\]

taken round the contour \( A(-i\pi/t)B(i\pi/t)C(R+i\pi/t)D(R-i\pi/t), \)
and let \( I_{P,R'} \) be the same integral taken round the contour \( A(-i\pi/t) \cdot B(i\pi/t)E(-R'+i\pi/t)F(-R'-i\pi/t). \) We shall assume now that \( P > R \) and \( P > R', \) and that both contours are indented so that (supposing that \( R \) and \( R' \) are integers) the first \( R \) of every ascending sequence of poles of \( \prod P(s) \) fall inside \( ABCD, \) and the first \( R' \) of every descending sequence of poles of \( \prod P(s) \) fall inside \( ABEF. \) We assume also that \( q = e^{-t}, t > 0, \) though the restriction that \( q \) is real can easily be removed from the final result by analytic continuation over the circle \( |q| < 1. \)

By the periodicity of the integrand, we have

\[
\int_{BC} \prod (s)ds = \int_{AD} \prod (s)ds \quad \text{and} \quad \int_{FA} \prod (s)ds = \int_{EB} \prod (s)ds.
\]
Hence
\[ I_{P, R} = \int_{AB} \prod_{s} ds + \int_{CD} \prod_{s} ds, \]
and
\[-I_{P, R'} = \int_{AB} \prod_{s} ds + \int_{EF} \prod_{s} ds. \]

But
\[ I_{P, R} = \sum \text{(residues within } A B C D \text{ of } \prod_{s} ds), \]
\[ = \frac{1}{t} \sum_{\mu=1}^{D} \prod_{s} \left[ (a) + d_{\mu}, (b) - d_{\mu}, 1 - z + d_{\mu}, z - d_{\mu} \right] (c) + d_{\mu}, (d') - d_{\mu}, 1 \]
\[ \times \sum_{n=0}^{R} \frac{((c) + d_{\mu})_{n}(1 + d_{\mu} - (b))_{n} Q^{n}}{((a) + d_{\mu})_{n}(1 + d_{\mu} - (d))_{n}}, \]
where \( Q = (-q^{(n+1)/2+d_{\mu}})^{(D-B)}q^{b_{1}+\cdots+b_{B}-d_{1}-\cdots-d_{D}+z}, \) that is
\[ I_{P, R} = \sum_{t}^{D} \prod_{s}^{P} \sum_{c}^{R} (d), \text{ say.} \]

Similarly,
\[-I_{P, R'} = \frac{1}{t} \sum_{\nu=1}^{C} \prod_{s} \left[ (b) + c_{\nu}, (a) - c_{\nu}, z + c_{\nu}, 1 - z - c_{\nu} \right] (d) + c_{\nu}, (c') - c_{\nu}, 1 \]
\[ \times \sum_{n=0}^{R'} \frac{((d) + c_{\nu})_{n}(1 + c_{\nu} - (a))_{n} Q'^{n}}{((b) + c_{\nu})_{n}(1 + c_{\nu} - (c))_{n}}, \]
where \( Q' = (-q^{(n+1)/2+c_{\nu}})^{(C-A)}q^{a_{1}+\cdots+a_{A}-c_{1}-\cdots-c_{C}+1-z}, \) that is,
\[-I_{P, R'} = \sum_{t}^{C} \prod_{s}^{P} \sum_{c}^{R'} (c), \text{ say,} \]

and so
\[
\int_{AB} \prod_{s} ds = \sum_{t}^{D} \prod_{s}^{P} \sum_{c}^{R} (d) + \int_{DC} \prod_{s} ds
\]

\[= \sum_{t}^{C} \prod_{s}^{P} \sum_{c}^{R'} (c) + \int_{EF} \prod_{s} ds. \]

Now
\[
\left| \int_{DC} \prod_{s} ds \right| \leq \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} \left| \prod_{s} (R + iv) \right| dv,
\]
and

\[ \left| \int_{R_{P}} \prod_{P} (s) ds \right| \leq \frac{1}{2\pi} \int_{-\pi/i}^{\pi/i} \left| \prod_{R'} (-R' + ir) \right| dr. \]

It has been shown previously (Slater [3]) that if \( D = B \) and \( \Re \sum (b - d) > 0 \), or if \( D > B \), \( \int_{AB} \prod_{P} (s) ds = \sum_{D} \prod_{P} \sum_{-\infty}^{\infty} (d) \). Also if \( A = C \) and \( \Re \sum (a - c) > 0 \), or if \( C > A \),

\[ \int_{AB} \prod_{P} (s) ds = \sum_{C} \prod_{P} \sum_{-\infty}^{\infty} (c). \]

In all cases, even when \( C < A \), or when \( D < B \), we have for \( R \) fixed

\[ \left| \int_{D_{C}} \prod_{P} (s) ds \right| \leq \frac{1}{t} \int_{-\infty}^{\infty} \left| \frac{1 + q^{(a) + R + n}}{1 + q^{1 - z + R + n}} \right| \left| \frac{1 + q^{(b) - R + n}}{1 - q^{(c) + R + n}} \right| \left| \frac{1 + q^{s - R + n}}{1 - q^{(d) - R + n}} \right|. \]

But the next term of the series \( \sum_{D} \prod_{P} \sum_{R} (d) \) would be

\[ \sum_{\mu = 1}^{D} \prod_{\mu} \left[ \begin{array}{c} (a) + d_{\mu}, 1 - z + d_{\mu}, (b) - d_{\mu}, z - d_{\mu} \\ (c) + d_{\mu}, (d)' - d_{\mu}, 1 \end{array} \right] \times \frac{((c) + d_{\mu})_{R + 1}(1 + d_{\mu} - (b))_{R + 1}Q_{R + 1}}{((a) + d_{\mu})_{R + 1}(1 + d_{\mu} - (d))_{R + 1}} \]

which is of the same order in \( R \) as \( \int_{D_{C}} \prod_{P} (s) ds \). Similarly, for \( R' \) fixed \( \int_{P_{E}} \prod_{P} (s) ds \) is also bounded above as \( P \to \infty \), and this integral is of the same order in \( R' \) as the \( (R' + 1) \)th term of the series \( \sum_{C} \prod_{P} \sum_{R'} (c) \).

Hence we have

\[ \frac{1}{2\pi i} \int_{-\pi/i}^{\pi/i} \prod_{1}^{\infty} \left[ \begin{array}{c} (a) + s, 1 - z + s, (b) - s, z - s; (c) + s, (d) - s \end{array} \right] ds
\]

\[ \sim \frac{1}{t} \sum_{\mu = 1}^{D} \prod_{\mu} \left[ \begin{array}{c} (a) + d_{\mu}, 1 - z + d_{\mu}, (b) - d_{\mu}, z - d_{\mu} \\ (c) + d_{\mu}, (d)' - d_{\mu}, 1 \end{array} \right] \times B + C \Phi_{A + D - 1} \left[ \begin{array}{c} (c) + d_{\mu}, 1 + d_{\mu} - (b); \\ (a) + d_{\mu}, 1 + d_{\mu} - (d)'; \end{array} Q \right] \]

\[ \sim \frac{1}{t} \sum_{c = 1}^{C} \prod_{c = 1}^{\infty} \left[ \begin{array}{c} (b) + c_{r}, (a) - c_{r}, z + c_{r}, 1 - z - c_{r} \\ (d) + c_{r}, (c)' - c_{r}, 1 \end{array} \right] \times A + D \Phi_{B + C - 1} \left[ \begin{array}{c} (d) + c_{r}, 1 + c_{r} - (a); \\ (b) + c_{r}, 1 + c_{r} - (c)'; \end{array} Q' \right] \]

(2.2)
where
\[ Q = (-q^{(n+1)/2+d_{\mu}}(D-B)q^{b_1+\cdots+b_{D-1}}d_{D+1}, \]
and
\[ Q' = (-q^{(n+1)/2+c_{\nu}}(C-A)q^{a_1+\cdots+a_{C-1}}c_{C+1}}, \]
even when \( D\leq B \) or when \( C\leq A \).

In particular, let
\[ A = B = 0, \quad C = 1, \quad D = 2, \]
then
\[
\frac{1}{2\pi i} \int_{-\pi/2}^{\pi/2} \pi \left[ 1 - x + s, x - s; a + s, 1 - b - s, -s \right] ds
\]
\[
= \prod_{a, 1 - b, 1} \left[ 1 - x, x \right] _1 \Phi_1 \left[ a; b; -q^{(n+1)/2+b-1} \right]
\]
\[
+ \prod_{1 + a - b, b - 1, 1} \left[ 2 - b - x, x - 1 + b \right] \times \left[ 1 + a - b; 2 - b; -q^{(n+1)/2+x} \right]
\]
\[
\times \left[ 1 + a - b, a, 1 \right] _2 \Phi_0 \left[ 1 + a - b, a; q^{1-x-a} \right]
\]
(2.3)

and, if \( A = B = C = D, B = 0 \), then
\[
\frac{1}{2\pi i} \int_{-\pi/2}^{\pi/2} \pi \left[ b + s, 1 - x + s, x - s; a + s, -s \right] ds
\]
\[
= \prod_{a, 1} \left[ b, 1 - x, x \right] _1 \Phi_1 \left[ a; b; q^x \right]
\]
\[
\times \left[ 1 + a - b, a, 1 \right] _2 \Phi_0 \left[ a, 1 + a - b; -q^{1+b-2a-x-(n+1)/2} \right].
\]
(2.4)

But
\[
_1 \Phi_1 \left[ a; b; q^{(n+1)/2+x} \right]
\]
(2.5)
\[
= \prod \left[ 1 + a - b + x + \pi i/t \right] _1 \Phi_1 \left[ b - a; b; -q^{1+a+x-b} \right]
\]
and
\[
_1 \Phi_1 \left[ a; b; q^x \right] = \sqrt{\prod \left[ x \right] _1 \Phi_1 \left[ b - a; b; -q^{(n+1)/2+x+a-1} \right].}
\]
(2.6)
Hence we have

\[ 1 \Phi_1[a; b; q^{(n+1)/2+x}] \]
\[ \sim \prod_{n=0}^{\infty} \left[ 1 + a - b + x + \pi i/t, b - a - x + \pi i/t, 1 - b \right] \]
\[ \times 2 \Phi_0[a, 1 + a - b; -q^{1-2a+b-x}]
\]
\[ + \prod_{n=0}^{\infty} \left[ x + 1 + \pi i/t, \pi i/t - x, a \right] \]
\[ \times 2 \Phi_0[1 - a, b - a; q^{1-(n+1)/2}] \]
\[ (2.7) \]

and

\[ 1 \Phi_1[a; b; q^{x}] \]
\[ \sim \prod_{n=0}^{\infty} \left[ 2a - 1 - x, 1 - b \right] \]
\[ \times 2 \Phi_0[1 - a, b - a; q^{1-2a+b-x}]
\]
\[ + \prod_{n=0}^{\infty} \left[ a + x, 1 - a - x, b - a \right] \]
\[ \times 2 \Phi_0[1 + a - b, a; -q^{1-2a+b-x-(n+1)/2}] \]
\[ (2.8) \]

These two results are the analogues for basic series of (1.4) above.

**REFERENCES**


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