1. In the differential equation
\[ x'' + (\lambda - f) x = 0, \]
let \( f = f(t) \) be a real-valued, continuous function on \( 0 \leq t < \infty \) and suppose that \( \lambda \) is a real parameter. If (1) is of the limit-point type, then (1) and a boundary condition of the type
\[ x(0) \cos \alpha + x'(0) \sin \alpha = 0, \quad 0 \leq \alpha < \pi, \]
determine, for every fixed \( \alpha \), a boundary value problem on \( 0 \leq t < \infty \) with a spectrum (of \( \lambda \)-values) \( S = S_\alpha \) [7]. It is known that the set \( S' \) consisting of the set of cluster points of \( S_\alpha \) is independent of \( \alpha \); loc. cit. p. 251. The following theorem will be proved:

(*) If \( f(t) \) denotes a real-valued, continuous function on the half-line \( 0 \leq t < \infty \) satisfying the condition
\[ \int_0^\infty |f(t)| \, dt \text{ converges} \quad \left( \int_0^\infty = \lim_{T \to \infty} \int_0^T \right), \]
then (1) is of the limit-point type and
\[ S' = [0, \infty). \]

It is noteworthy that (3) may exist only conditionally and that
\[ \int_0^\infty |f(t)| \, dt < \infty \]
is not assumed. Actually, if (3') holds, much more is known. In fact, in this case, there exist asymptotic formulas for the solutions of (1) when \( \lambda > 0 \) ([8, p. 421]; cf. also [7, p. 258], in case \( f(t) \to 0 \) as \( t \to \infty \)) which guarantee, in particular, that \( 0 \leq \lambda < \infty \) is in the continuous spectrum for every boundary value problem determined by (1) and (2). Obviously, the requirement (3) is compatible with \( T^{-1} \int_0^T |f(t)| \, dt \to \infty \), as \( T \to \infty \), and, in fact, even with the requirement that \( \int_0^\infty |f(t)| \, dt \to \infty \) arbitrarily fast. Thus, if \( \phi(t) \) denotes any positive function satisfying \( \phi(t) \to \infty \) as \( t \to \infty \), there exists a continuous function \( f(t) \) on \( 0 \leq t < \infty \) satisfying (3) and \( \phi(T) = o\left(\int_0^T |f(t)| \, dt\right) \), as \( T \to \infty \).
On the other hand, most of the criteria for (4) or $S' \supset [0, \infty)$ involve $|f(t)|$ rather than $f(t)$, and, as a consequence, require that $f(t)$ be close to zero "on the average." For instance, it is known that

\[(5) \quad T^{-1} \int_0^T |f(t)| \, dt \to 0, \quad T \to \infty,\]

is enough to guarantee that $S' \supset [0, \infty)$, although

$$\lim \sup T^{-1} \int_0^T |f(t)| \, dt < \infty$$

is not; cf. [3, p. 80]. Moreover, (5) is compatible with $S' = (-\infty, \infty)$; cf. [3].

2. Proof of (*). Since $f$ satisfies (3), it is clear that $\int_0^T (\lambda - f(t)) \, dt \to \infty$ as $T \to \infty$ whenever $\lambda > 0$. It follows that (1) is oscillatory (i.e., every nontrivial solution possesses an infinity of zeros clustering at $+\infty$) whenever $\lambda > 0$; [10], cf. also [4]. Next, it will be shown that, in view of (3), the equation (1) is nonoscillatory whenever $\lambda < 0$. (It is of interest to note here that there are known necessary and sufficient conditions in order that an equation (1) be oscillatory; cf., e.g., [5; 9]. In the present case it will be convenient for later use to give the direct argument below.)

Suppose first that $\lambda$ is arbitrary and that (1) possesses an oscillatory solution $x=x(t)(\neq 0)$ with zeros tending to infinity. If $S < T$ denote two zeros of $x(t)$, a multiplication of (1) by $x$ followed by an integration leads to

\[(6) \quad \int_S^T x'' \, dt = \lambda \int_S^T x^2 \, dt - \int_S^T f x^2 \, dt.\]

An integration by parts of the second integral on the right side of the equation (6) yields

\[(7) \quad \int_S^T f x^2 \, dt = -2 \int_S^T x x' F(t) \, dt, \quad F(t) = \int_0^t f(s) \, ds.\]

In view of (3), $F(t) = \text{const.} + o(1)$ as $t \to \infty$, and an application of the Schwarz inequality to the second integral of (7) now implies

\[\int_S^T x'' \, dt = \lambda \int_S^T x^2 \, dt + o \left( \int_S^T x^2 \, dt \int_S^T x'' \, dt \right)^{1/2},\]

and hence,

\[(8) \quad A = \lambda + o(A^{1/2}), \quad \text{where} \quad A = \int_S^T x'' \, dt / \int_S^T x^2 \, dt,\]
where the "o term" refers to $S \to \infty$. It readily follows from (8) that $\lambda \geq 0$ and so (1) must be nonoscillatory whenever $\lambda < 0$.

It follows from the last result that (1) is of the limit-point type and that, in addition, $S' \subset [0, \infty)$; [1], cf. also [2]. There remains to be shown that the half-line $\lambda \geq 0$ belongs to $S'$. To this end, consider any boundary condition (2) for a fixed value $\alpha$ and let

$$m_\alpha(\lambda) = \min | \lambda - \mu |,$$

when $\mu$ is in the (closed) set $S_\alpha$. It will be shown that

$$(9)\quad m_\alpha(\lambda) \equiv 0 \quad \text{for } \lambda > 0 \quad \text{(hence for } \lambda \geq 0),$$

and so (4) will follow.

Let $g = g(t)$ denote any function of class $C^2$ on the finite interval $0 \leq t \leq T$ and satisfying the boundary conditions (2) and

$$(10)\quad g(T) = g'(T) = 0.$$

Then the argument of [6, pp. 579–580] shows that

$$(11)\quad m_\alpha^2(\lambda) \int_0^T g^2 dt \leq \int_0^T (L(g) + \lambda g)^2 dt \quad (L(x) \equiv x'' - f x).$$

Next, let $\mu$ and $\epsilon$ be positive and suppose that $g(t) = y(t)h(t)$, where $h(t) = \cos (\mu^{1/2}t)$ and $y(t)$ is a nontrivial (oscillatory) solution of (1) for $\lambda = \epsilon$, so that $L(y) + \epsilon y = 0$, and satisfying (2) for $x = y$. Next, let $T$ be chosen so that

$$(12)\quad y(T) = 0.$$

In addition, since (1) is of the limit-point type, the number $\epsilon$ can be chosen arbitrarily small and so that the function $y$ satisfies

$$(13)\quad \int_0^\infty y^2 dt = \infty ;$$

cf. [7]. It will be supposed that $\mu = \mu(T)$ is chosen so that

$$(14)\quad \cos (\mu^{1/2}T) = 0;$$

hence, as a consequence of (12) and the relation $g' = y'h + yh'$, $g(t)$ also satisfies (10). In view of

$$(15)\quad L(g) + \lambda g = (\lambda - \mu - \epsilon) y h + 2 y' h',$$

the relation (11) and the inequality $(a+b)^2 \leq 2(a^2+b^2)$ now yield

$$(16)\quad m_\alpha^2(\lambda) \int_0^T h^2 y^2 dt \leq \text{const.} \int_0^T [\mu y'^2 + (\lambda - \mu - \epsilon)^2 y^2] dt.$$

Next, let $T = T_1 < T_2 < \cdots$ denote the positive zeros of $y = y(t)$
and choose $\mu_n = \mu(T_n)$ (hence $h = h_n$) so that (14) holds for $T = T_n$ and $\mu_n \rightarrow \lambda (>0)$. (That this can be done is clear.) It follows from (16) that, as $n \rightarrow \infty$,

$$m^2_n(\lambda) \leq \text{const. lim sup} \left[ \int_0^{T_n} (\epsilon^2 y^2 + \lambda y'^2) dt / \int_0^{T_n} h_n y^2 dt \right].$$

A calculation like that of [6, p. 581], together with (12), yields

$$\int_0^{T_n} h_n y^2 dt \geq \frac{1}{2} \int_0^{T_n} y^2 dt - \frac{1}{2} \mu_n^{-1/2} \left( \int_0^{T_n} y'^2 dt \int_0^{T_n} y'^2 dt \right)^{1/2}.$$

If use is made of (13), a calculation similar to that used in obtaining (8) shows that $A = \epsilon + o(A^{1/2})$, as $T_n \rightarrow \infty$, where

$$A = A_n = \int_0^{T_n} y'^2 dt / \int_0^{T_n} y^2 dt.$$

This implies however that $A(T_n) < \text{const. } \epsilon$ for $T_n$ large, and hence, by (18), $\int_0^{T_n} h_n y^2 dt \geq \text{const. } \int_0^{T_n} y^2 dt > 0$ for $T_n$ large and for a sufficiently small $\epsilon$. Finally, relation (17) now implies $m^2(\lambda) \leq \text{const. } (\epsilon^2 + \epsilon \lambda)$. Since $\epsilon > 0$ can be chosen arbitrarily small, relation (9) follows and the proof of (*) is complete.

References