Borel-Cantelli lemma and of (10) where λ is replaced by λ/μ. This finishes the proof.

It is interesting to note that although λ appears in the definition of X_{nk} and in (3), it does not appear in (11) or (14).

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FAMILIES OF PERIODIC SOLUTIONS OF SYSTEMS HAVING RELATIVELY INVARIANT LINE INTEGRALS[1]

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The purpose of this paper is to generalize to all systems having relatively invariant line integrals a known theorem about Lagrangian (or Hamiltonian) systems of differential equations. This known theorem is roughly to the effect that in any given family of periodic solutions the period is at most a function of the energy constant alone; that is, it does not depend upon the parameters of the family except in so far as the energy is a function of these parameters. An exception may occur, however, if the energy itself is independent of the parameters.

The history of this particularly elegant theorem began over eighty years ago, but the only presentation in a standard treatise known to me is in the book of A. Wintner, *Analytical foundations of celestial mechanics*, Princeton University Press, 1941, pp. 73, 74, and p. 414, where reference to the older literature is to be found. The only flaw in this treatment is the omission of mention of the exception noted above in case the energy constant is independent of the parameters. A satisfactory treatment in this respect is given by G. Herglotz, among the collection of papers by various authors published in book form: *Probleme der Astronomie; Festschrift für Hugo v. Seeliger*, Berlin, 1924, pp. 197–199.

The generalization given in this paper (cf. below the statement of the main theorem) is a bit more than a mere extension of the classical theorem to the Pfaffian equations of Birkhoff (which, by a suitable transformation on the dependent variables, can be written in Hamiltonian form, at least in the neighborhood of a given periodic solu-

tion); for we make no assumption about the rank of the skew-symmetric matrix $(\partial A_i/\partial x_j - \partial A_j/\partial x_i)$. The order of the system is not even assumed to be even.

In order to summarize certain preliminary facts, we begin with a statement of the following known lemmas:

LEMMA 1. *Consider the system*

(1) $$dx_i/dt = X_i(x_1, \cdots, x_n), \qquad i = 1, \cdots, n,$$

and the line integral

(2) $$\int_C \sum_{i=1}^{n} A_i(x_1, \cdots, x_n)dx_i$$

in which the X's are of class C' and the A's of class C'' in a domain R of n-space. A necessary and sufficient condition that (1) admit (2) as a relatively invariant line integral (in the sense of Poincaré) is that there should exist a function H of class C'' in this domain such that

(3) $$\sum_{j=1}^{n} (\partial A_i/\partial x_j - \partial A_j/\partial x_i)X_j = \partial H/\partial x_i,$$

identically in R.

(Cf. E. Goursat, *Leçons sur le problème de Pfaff*, Paris, 1922, p. 219; but the understanding reader can probably construct his own proof far more easily than by deriving the lemma as a special case of various general results in the literature. Actually the lemma is not essential for the rest of the paper except to show the significance of (3), which is henceforth assumed.)

LEMMA 2. *If H is any function satisfying (3), then $H(x_1, \cdots, x_n)$ is a first integral of (1).*

For $\sum_i(\partial H/\partial x_i)X_i = \sum_{i,j}(\partial A_i/\partial x_j - \partial A_j/\partial x_i)X_iX_j \equiv 0$ because of the skew-symmetry of $(\partial A_i/\partial x_j - \partial A_j/\partial x_i)$.

We now state our main result:

THEOREM. *We assume again that the X's are of class C', that the A's and H are of class C'', and that (3) holds. We also assume that we have a family of solutions of (1) of class C'' depending on m parameters $c_1, \cdots, c_m$, and that these solutions are periodic with a period $T = T(c_1, \cdots, c_m) = T(c)$ which, considered as a function of the parameters, is of class C'. We also denote by $h(c_1, \cdots, c_m) = h(c)$ the value of the first integral H for the solution with parameter values (c). Then all the jacobians*

(4)
$$\frac{\partial(h, T)}{\partial(c_k, c_h)} = 0, \qquad h, k = 1, 2, \cdots, m,$$

identically.

To prove this, the family of solutions is represented in the form

$$(5) \quad x_i = x_i(t, c_1, \cdots, c_m) = x_i(t, c) = x_i(t + T(c), c), \quad i = 1, \cdots, n.$$

Setting $t = 0$ and differentiating both sides of the last equality with respect to c_k, we find that

$$(6) \qquad \frac{\partial x_i}{\partial c_k}\bigg|_{t=0} = \dot{x}_i \frac{\partial T}{\partial c_k}\bigg|_{t=T(c)} + \frac{\partial x_i}{\partial c_k}\bigg|_{t=T(c)}.$$

Here, of course, $\dot{x}_i$ refers to the derivative of $x_i(t, c)$ with respect to t. Let $S = S(c) = \int_0^{T(c)} (\sum_{i=1}^n A_i \dot{x}_i + H) dt$, in which the arguments of the A's and H are understood to be given by (5). Differentiating this last we find that

$$
\begin{aligned}
\frac{\partial S}{\partial c_k} = &\left(\sum A_i \dot{x}_i + H \right) \frac{\partial T}{\partial c_k}\bigg|_{t=T(c)} \\
(7) \\
&+ \int_0^{T(c)} \left(\sum_i A_i \frac{\partial \dot{x}_i}{\partial c_k} + \sum_{i,j} \frac{\partial A_i}{\partial x_j} \frac{\partial x_j}{\partial c_k} \dot{x}_i + \sum_j \frac{\partial H}{\partial x_j} \frac{\partial x_j}{\partial c_k} \right) dt.
\end{aligned}
$$

The first terms in the integrand may be integrated by parts:

$$
\begin{aligned}
\int_0^{T(c)} \left(\sum_i A_i \frac{\partial \dot{x}_i}{\partial c_k} \right) dt = &\sum_i A_i \frac{\partial x_i}{\partial c_k}\bigg|_{t=T(c)} - \sum_i A_i \frac{\partial x_i}{\partial c_k}\bigg|_{t=0} \\
&- \int_0^{T(c)} \left(\sum_{i,j} \frac{\partial A_i}{\partial x_j} \dot{x}_j \frac{\partial x_i}{\partial c_k} \right) dt.
\end{aligned}
$$

We now interchange the indices i and j in the double summation in the last term, and we also reduce the first two terms with the help of (6) and the fact that $A_i|_{t=T(c)} = A_i|_{t=0}$ from the given periodicity of the family. We thus obtain

$$
\begin{aligned}
\int_0^{T(c)} \left(\sum_i A_i \frac{\partial \dot{x}_i}{\partial c_k} \right) dt = &- \sum_i A_i \dot{x}_i \frac{\partial T}{\partial c_k}\bigg|_{t=T(c)} \\
&- \int_0^{T(c)} \left(\sum_{i,j} \frac{\partial A_j}{\partial x_i} \dot{x}_i \frac{\partial x_j}{\partial c_k} \right) dt.
\end{aligned}
$$

Using this result in (7) and also setting $\dot{x}_i = X_i$, since (1) is satisfied, we find that

$$\frac{\partial S}{\partial c_k} = H \frac{\partial T}{\partial c_k}\Big|_{t=T(c)}$$

$$+ \int_0^{T(c)} \left(\sum_j \left(\sum_i \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right) X_i + \frac{\partial H}{\partial x_j} \right) \frac{\partial x_j}{\partial c_k} \right) dt.$$

Hence, finally from (3), Lemma 2, and the definition of h, we have the very simple formula,

$$(8) \qquad\qquad\qquad \frac{\partial S}{\partial c_k} = h(c) \frac{\partial T}{\partial c_k}, \qquad\qquad k = 1, 2, \cdots, m.$$

If T were known to be of class C'' the same would be true of S and the fact that $\partial^2 S/\partial c_k \partial c_l = \partial^2 S/\partial c_l \partial c_k$ would lead at once to (4).

Since, however, we assume only that T is of class C', we arrive at the same result by the device of letting $W = hT - S$, so that $\partial W/\partial c_k = (\partial h/\partial c_k)T + h(\partial T/\partial c_k) - \partial S/\partial c_k = T(\partial h/\partial c_k)$ by (8). Now h is known, of course, to be of class C'', so the same is true of W, and we arrive at (4) from the fact that $\partial^2 W/\partial c_k \partial c_l = \partial^2 W/\partial c_l \partial c_k$.

In the theorem just proved we have deliberately left our conclusion in the form (4), because of the difficulty of drawing any completely general conclusions on functional dependence from it. Several not mutually exclusive cases may be considered.

CASE 1. *If not all of the derivatives of h vanish at a point (c^0), then there is a function ϕ of one variable of class C' such that $T(c) \equiv \phi(h(c))$ in the neighborhood of (c^0).* This may be regarded as the general case and, for Hamiltonian systems, it is the basis of the rough statement in our first paragraph.

CASE 2. *If not all of the derivatives of T vanish at a point (c^0), then there is a function ψ of one variable of class C' such that $h(c) \equiv \psi(T(c))$ in the neighborhood of (c^0).* This may also be regarded as the general case; but that it is not coextensive with Case 1, even for Hamiltonian systems, will be shown below by means of examples.

CASE 3. *If all the derivatives of both T and h vanish near (c^0) identically, then, of course, both T and h are constants*, and we have no difficulty in writing either $T = \phi(h)$ or $h = \psi(T)$, or more inclusively

$$(9) \qquad\qquad G(T, h) = 0, \qquad |G_T| + |G_h| \neq 0.$$

CASE 4. *If all the derivatives of both T and h vanish at (c^0) but not identically in the neighborhood of (c^0)*, we have a situation shunned by most elementary writers on functional dependence. If we assume that T and h are *analytic*, we can still assert the validity of (9) for a suitably chosen G (cf. G. A. Bliss, Amer. Math. Soc. Colloquium

Publications, vol. 3_1, 1913, pp. 67–70). Otherwise the situation is obscure.

We close with mention of a few examples of systems in which T varies while h remains fixed, thus showing that Case 2 can occur without Case 1.

Any equilibrium point of any Hamiltonian system may be regarded as a family of periodic solutions having an arbitrary varying period, all with the same energy. But this example is so highly degenerate as to suggest the possibility of ruling out the exhibited phenomenon by some such requirement as that the functions $x_i(t, c)$ of (5) should not reduce to mere constants.

Herglotz (loc. cit.) gives the example afforded by the motion of a particle attracted toward a fixed center by a force inversely proportional to the *cube* of its distance from the center. In this system the circular solutions form an isoenergetic family with varying period. He shows that this is the only possibility of this phenomenon in central force problems.

Other nontrivial examples, having nothing to do with the central force problem, have been devised but, for brevity, are here omitted.

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