A BASIC SET OF HOMOGENEOUS HARMONIC POLYNOMIALS IN \( k \) VARIABLES

E. P. MILES, JR. AND ERNEST WILLIAMS

1. The authors present basic sets of:
   (a) homogeneous harmonic polynomials of degree \( n \) in \( k \) variables, \( k \geq 3 \);
   (b) associated polynomial solutions of the wave equation, and
   (c) analogous solutions for \( \sum_{j=1}^{k} (\partial^{s}u)/(\partial x_{j}^{s})=0, \ s=3, 4, \ldots \).

2. For any set of non-negative integers \((b_{j})\) such that \( b_{1} \leq 1 \) and \( \sum_{j=1}^{k} b_{j}=n \), let

\[
H_{b_{1}b_{2}\ldots b_{k}}^{n}(x_{1}, x_{2}, \ldots, x_{k})
\]

\[
= \sum (-1)^{[a_{1}/2]} \frac{n!}{\prod_{j=1}^{k} a_{j}!} \prod_{j=2}^{k} \left( \frac{(b_{j}-a_{j})!}{2} \right)! \prod_{j=1}^{k} x_{j}^{a_{j}}
\]

where the summation is extended over all \((a_{j})\) such that:

1. \( a_{j} \equiv b_{j} \mod 2, \ j=1, 2, \ldots, k, \)
2. \( \sum_{j=1}^{k} a_{j}=n, \)
3. \( a_{j} \leq b_{j}, \ j=2, 3, \ldots, k. \)

The polynomials (1) form a basic set of homogeneous harmonic polynomials in \( k \) variables. The proof is given in three parts.

A. The polynomials (1) are linearly independent since each contains exactly one different nonvanishing term of the monomials \( x_{1}^{b_{1}}x_{2}^{b_{2}} \cdots x_{k}^{b_{k}}, \sum_{j=1}^{k} a_{j}=n, a_{1} \leq 1. \)

Moreover, since the number of terms in \((\sum_{j=1}^{k} x_{j})^{n}\) is

\[
\binom{n+k-1}{k-1},
\]

the total number of monomials of type \( x_{2}^{a_{2}}x_{3}^{a_{3}} \cdots x_{k}^{a_{k}}, \sum_{j=2}^{k} a_{j}=n, \) and of type \( x_{1}x_{3}^{a_{3}}x_{3}^{a_{3}} \cdots x_{k}^{a_{k}}, \sum_{j=2}^{k} a_{j}=n-1, \) is

\[
\binom{n+k-2}{k-2} + \binom{n+k-3}{k-2} = \binom{n+k-3}{k-3} \left( \frac{2n}{k-2} + 1 \right).
\]

Thus the polynomials (1) are

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\[
\binom{n + k - 3}{k - 3} \left( \frac{2n}{k - 2} + 1 \right)
\]
in number.

B. They are harmonic. Let \( c_j, j = 1, \cdots, k \), be such that

1. \( c_j \equiv b \mod 2 \),
2. \( c_j \leq b_j, j = 2, 3, \cdots, k \), and
3. \( \sum_{j=1}^{k} c_j = n - 2 \).

The coefficient \( B_{c_1, c_2, \ldots, c_k} \) of \( \prod_{j=1}^{k} x_j^{c_j} \) in \( \nabla^2 H_{b_1, \ldots, b_k}(x_1, \ldots, x_k) \) is given by

\[
B_{c_1, \ldots, c_k} = (-1)^{\lfloor c_1/2 \rfloor + 1} \frac{n!}{\prod_{j=1}^{k} c_j!} \left[ \frac{c_1}{2} + 1 \right]! - \sum_{j=2}^{k} \frac{c_1}{2}! \left( \frac{b_j - c_j}{2} \right)! \frac{n!}{\prod_{j=1}^{k} \left( b_j - c_j \right)!} \left[ \frac{c_1}{2} + 1 - \frac{1}{2} \sum_{j=2}^{k} b_j + \frac{1}{2} \sum_{j=2}^{k} c_j \right] \]

\[
= (-1)^{\lfloor c_1/2 \rfloor + 1} \frac{n! \left[ \frac{c_1}{2} \right]!}{\prod_{j=1}^{k} c_j! \prod_{j=2}^{k} \left( b_j - c_j \right)!} \left[ \frac{c_1}{2} + 1 - \frac{1}{2} (n - b_1) + \frac{1}{2} (n - 2 - c_1) \right] \]

\[
= (-1)^{\lfloor c_1/2 \rfloor + 1} \frac{n! \left[ \frac{c_1}{2} \right]!}{\prod_{j=1}^{k} c_j! \prod_{j=2}^{k} \left( b_j - c_j \right)!} \left[ \frac{c_1}{2} - \frac{1}{2} (c_1 - b_1) \right] = 0.
\]
C. For a general homogeneous polynomial $H^n_k$ of degree $n$ in $k$ variables the vanishing of the Laplacian $\nabla^2 H^n_k$ provides

\[
\binom{n + k - 3}{k - 1}
\]
equations on the \(
\binom{n + k - 1}{k - 1}
\) coefficients of $H^n_k$. Thus the number of linearly independent homogeneous harmonic polynomials of degree $n$ in $k$ variables is

\[
\binom{n + k - 1}{k - 1} - \binom{n + k - 3}{k - 1} = \binom{n + k - 3}{k - 3} \left( \frac{2n}{k - 2} + 1 \right),
\]

which is the number of polynomials (1).

3. It is worth noting that the polynomials obtained from (1) by deleting the factor $(-1)^{[a_1/s]}$ are solutions of the generalized wave equation $\sum_{j=2}^{k} \partial^2 u/\partial x_j^2 = \partial^2 u/\partial x_1^2$, which form a basic set for that equation.

4. Further, for each set of $k$ non-negative integers $b_j$ such that $\sum_{j=1}^{k} b_j = n$, $b_1 \leq s - 1$, the polynomials

\[
H^{n,s}_{b_1, b_2, \ldots, b_k}(x_1, x_2, \ldots, x_k)
\]

\[
= \sum (-1)^{[a_1/s]} \frac{n!}{\prod_{j=1}^{k} a_j!} \prod_{j=2}^{k} \left( \frac{b_j - a_j}{s} \right) \prod_{j=1}^{k} x_j^{a_j},
\]

where the summation extends over all $a_j$ such that

(1) $a_j \equiv b_j \mod s$, $j = 1, 2, \ldots, k$,

(2) $\sum_{j=1}^{k} a_j = n$,

(3) $a_j \leq b_j$, $j = 2, 3, \ldots, k$,

provide a basic set of solutions for

\[
\sum_{j=1}^{k} \frac{\partial^s u}{\partial x_j^s} = 0.
\]

5. Of particular interest for harmonic polynomials is the case $k = 3$. Whittaker\(^1\) has obtained the general solution of $\nabla^2 U(x, y, z) = 0$ by means of an integral. Ketchum\(^2\) gives another form of the general solution as an analytic function of a hypervariable $w$ such that the $2n + 1$ linearly independent components of $w^n$ form a basic set of homogeneous harmonic polynomials of degree $n$. Both of these re-

\(^1\) Math. Ann. vol. 57 (1903) p. 333.
\(^2\) Amer. J. Math. vol. 51 (1929) p. 179.
results use trigonometric functions, and neither of them displays immediately a set of polynomial solutions of degree \( n \). Morse and Feshbach\(^3\) indicate how one obtains, from a special case of Whittaker's integral, a basic set of degree \( n \), but carry the computation only as far as \( n = 3 \). Courant and Hilbert\(^4\) give a basic set with only one of the \( 2n+1 \) members having real coefficients. The polynomials (1) for \( k = 3 \), \( x_1 = x \), \( x_2 = y \) and \( x_3 = z \), unlike those from the basic sets referred to above, are given explicitly for each \( n \). Thus, for \( n = 6 \) the 13 independent spherical harmonics are obtained by assigning \( (b_1 \ b_2 \ b_3) \) the values \( (0 \ 6 \ 0) \), \( (0 \ 5 \ 1) \), \( (0 \ 4 \ 2) \), \( (0 \ 3 \ 3) \), \( (0 \ 2 \ 4) \), \( (0 \ 1 \ 5) \), \( (0 \ 0 \ 6) \), \( (1 \ 5 \ 0) \), \( (1 \ 4 \ 1) \), \( (1 \ 3 \ 2) \), \( (1 \ 2 \ 3) \), \( (1 \ 1 \ 4) \), and \( (1 \ 0 \ 5) \) in turn. We display a typical member,

\[
H_{123}^6 = 60xyz^3 - 60xzy^3 - 20zx^3z + 12x^5z.
\]

6. The authors wish to express their appreciation to Professor Ernest Ikenberry who directed their attention to the 3-dimensional basic sets given by Morse-Feshbach and Courant-Hilbert and to the referee who pointed out the construction of basic sets of \( p+2 \) dimensions appearing in Higher transcendental functions,\(^5\) A. Erdélyi, editor. These sets for \( p+2 \) dimensions differ from those of the authors in that the coefficients are in general complex while those of the authors are real.

Added in proof. When these results for \( k = 3 \) only were presented at the International Congress of Mathematicians, September, 1954, Professors P. C. Rosenbloom and L. Bers kindly called the authors' attention to a three variable basic set of harmonic polynomials given by M. H. Protter [Generalized spherical harmonics, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 314–341]. In a forthcoming note in the Proceedings, the authors point out that their results for \( k = 3 \) give a single formulation for the four classes into which Protter's basic set was divided.

Alabama Polytechnic Institute