A THREE PARAMETER FAMILY OF PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS WITH TWO ASSOCIATE CLASSES

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1. **Introduction.** A null polarity in $PG(2h+1, p^n)\equiv S$ will be used to show the existence of a design $\mathcal{D}\equiv\mathcal{D}(s, h, p^n)$ $(s=1, 2, \cdots, h)$ with parameters

$$
\begin{aligned}
v &= q_{2h+1}, & b &= q_{2h+1}q_{2h-1}\cdots q_{2h-2s+1}/q_sq_{s-1}\cdots q_0,\\
r &= q_{2h-s}b/q_{2h+1}, & k &= q_{2h-1},\\
n_1 &= q_{2h}-1, & \lambda_1 &= q_{2h-s}q_{2h-s-1}b/q_{2h+1}q_{2h-1},\\
n_2 &= q_{2h+1}-q_{2h}, & \lambda_2 &= 0,
\end{aligned}
$$

$$
P_1 = (p^1_{jk}) = \begin{pmatrix} q_{2h-1}-2 & q_{2h}-q_{2h-1} \\ q_{2h}-q_{2h-1} & q_{2h+1}-2q_{2h}+q_{2h-1}\end{pmatrix},
$$

$$
P_2 = (p^2_{jk}) = \begin{pmatrix} q_{2h-1} & q_{2h}-q_{2h-1}-1 \\ q_{2h}-q_{2h-1}-1 & q_{2h+1}-2q_{2h}+q_{2h-1}\end{pmatrix},
$$

where $q_i=1+p^n+\cdots+p^{in}$. The special cases $\mathcal{D}(1, 1, 2)$ and $\mathcal{D}(1, 1, p^n)$ were derived by Bose and Shimamoto [**3**] and Clatworthy [**4**]. A method of constructing the design is indicated.

2. **Definitions.** A partially balanced incomplete block design with two associate classes having the parameters v, b, r, k, λ_i, n_i, p^i_{jk} $(i, j, k=1, 2)$ is an arrangement of v varieties into b blocks, together with a relationship called association defined for every pair of varieties satisfying the following conditions:

(i) Each of the v varieties occurs in r blocks. Each block contains k varieties and no variety appears more than once in a block.

(ii) Every pair of varieties are either first or second associates. Each variety has n_1 first associates and n_2 second associates.

(iii) Any pair of varieties which are ith associates occur together in exactly λ_i blocks $(i=1, 2)$.

(iv) For every pair of varieties which are ith associates, the number of varieties common to the jth associates of the first and the kth associates of the second in p^i_{jk} $(i, j, k=1, 2)$.

It has been shown by Bose and Nair [**2**] that the parameters p^i_{11}, n_i $(i=1, 2)$ determine the matrices P_i $(i=1, 2)$ uniquely. The following relations are proved:

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$$vr = bk, \tag{2.1}$$

$$v = n_1 + n_2 + 1, \tag{2.2}$$

$$\lambda_1 n_1 + \lambda_2 n_2 = r(k-1), \tag{2.3}$$

$$\overset{1}{p}_{12} = \overset{1}{p}_{21} = n_1 - \overset{1}{p}_{11} - 1, \tag{2.4}$$

$$\overset{1}{p}_{22} = n_2 - n_1 + \overset{1}{p}_{11} + 1, \tag{2.5}$$

$$\overset{2}{p}_{12} = \overset{2}{p}_{21} = n_1 - \overset{2}{p}_{11}, \tag{2.6}$$

$$\overset{2}{p}_{22} = n_2 - n_1 + \overset{2}{p}_{11} - 1. \tag{2.7}$$

A nonsingular null polarity Π defined in S is a 1-1 incidence preserving correspondence between the points and $2h$-spaces of S such that if x corresponds to the $2h$-space $\Pi(x)$, then $x \in \Pi(x)$. Let x_i $(i=0, 1, \cdots, s)$ be linearly independent points of an s-space $\Sigma \equiv \Sigma(x_0, x_1, \cdots, x_s) \subset S$. Then under the correspondence Π the s-space Σ corresponds to the $(2h-s)$-space $\Pi(x_0, x_1, \cdots, x_s) \equiv \Pi(x_0) \cap \Pi(x_1) \cap \cdots \cap \Pi(x_s)$. Two points $x, y \in S$ are called conjugate with respect to Π if $x \in \Pi(y)$. It follows that $y \in \Pi(x)$ and the relation is symmetric. In particular every point of S is self-conjugate with respect to Π. It is possible to find $s+1$ linearly independent elements $x_i \in S$ $(i=0, 1, \cdots, s)$, $s \leqq h$, which are conjugate in pairs. In fact we can choose $x_1 \in \Pi(x_0)$, $x_2 \in \Pi(x_0, x_1)$, $\cdots$, $x_s \in \Pi(x_0, x_1, \cdots, x_{s-1})$. In the future the notation $\Pi(x_0, x_1, \cdots, x_s)$ will indicate that the points x_i $(i=0, 1, \cdots, s)$ are linearly independent and conjugate in pairs with respect to Π.

3. **Existence of $\mathcal{D}(s, h, p^n)$.**

THEOREM. *Let the points of S be called varieties and let the $(2h-s)$-spaces $\Pi(x_0, x_1, \cdots, x_s)$ be called blocks. Let x, y be called first associates if they are conjugate with respect to Π and second associates otherwise. Then the varieties and blocks with the given definition of association may be abstractly identified with a design $\mathcal{D}(s, h, p^n)$ having the parameters stated in* §1.

We first observe that varieties and pairs of varieties of the same type are symmetric in S so that incidences are independent of the particular element or pair of elements of the same type.

The number of varieties of $\mathcal{D}$ is equal to the number of points of S, i.e. $v = q_{2h+1}$.

The number of blocks is the number of $(2h-s)$-spaces $\Pi(x_0, x_1, \cdots, x_s)$. The number of ways of choosing $s+1$ elements conjugate in pairs is

$$M = q_{2h+1}(q_{2h} - 1)(q_{2h-1} - q_1) \cdots (q_{2h-s+1} - q_{s-1})/(s+1)!.$$

Let $\Pi(B)$ be the s-space corresponding to any block B. Then B is the intersection of all $2h$-spaces $\Pi(y)$ with $y \in \Pi(B)$. In particular if y_i $(i=0, 1, \cdots, s)$ are linearly independent points of $\Pi(B)$ then $\Pi(y_0) \cap \Pi(y_1) \cap \cdots \cap \Pi(y_s) \equiv B$. The points y_i $(i=0, 1, \cdots, s)$ are conjugate in pairs since $\Pi(B) \subseteq B$ for $s \leqq h$. Thus B may be represented in the form $\Pi(y_0, y_1, \cdots, y_s)$ where y_i $(i=0, 1, \cdots, s)$ are any linearly independent points of $\Pi(B)$. It follows that in the enumeration $\Pi(x_0, x_1, \cdots, x_s)$ any $(2h-s)$-space is enumerated N times where

$$N = q_s(q_s - 1)(q_s - q_1) \cdots (q_s - q_{s-1})/(s+1)!.$$

Hence $b = M/N$.

The number of varieties in each block is $k = q_{2h-s}$ since this is the number of points in a $(2h-s)$-space. No variety appears more than once in a block.

The number of varieties conjugate to x is the number of points of $\Pi(x)$ different from x, i.e. $n_1 = q_{2h} - 1$.

If x, y are second associates, $y \in' \Pi(x)$ so that $y \in' \Pi(x) \cap \Pi(y)$. This implies $\lambda_2 = 0$.

Let x, y be points of S. The points common to the first associates of x and the first associates of y must be points of $E \equiv \Pi(x) \cap \Pi(y)$. If x, y are first associates they both belong to E so that $p_{11}^1 = q_{2h-1} - 2$. If x, y are second associates x, y do not belong to E so that $p_{11}^2 = q_{2h-1}$.

The remaining parameters follow directly from equations 2.1–2.7.

4. **Construction of the design.** All nonsingular null polarities are projectively equivalent so that the designs obtained from different polarities are equivalent. It is sufficient then to use any null polarity in the construction.

Let A be any nonsingular skew-symmetric $(2h+2) \times (2h+2)$ matrix with coefficients in $GF(p^n)$. In some allowable coordinate system any point x of $PG(2h+1 \cdot p^n)$ may be represented by a $(2h+2) \times 1$ matrix $X \neq 0$. $\rho X (\rho \in GF(p^n), \rho \neq 0)$ represents the same point or variety. The $2h$-space corresponding to X under the polarity Π defined by A is $U = AX$. $\rho U = A\rho X$ represents the same $2h$-space. The points y of U satisfy the equation $Y'AX = 0$ or $X'AY = 0$. Two points x, y represented by the matrices X, Y are thus conjugate whenever $X'AY = 0$. In particular $X'AX = 0$ since A is skew symmetric.

The sets $\Pi(x_0, x_1, \cdots, x_s)$ may be found inductively. The sets $\Pi(x_0)$ are the sets of points y satisfying the equation $Y'AX_0 = 0$ where x_0 is any point of S. Suppose the points $\Pi(x_0, x_1, \cdots, x_{s-1})$ have been constructed. Let x_s be any point of $\Pi(x_0, x_1, \cdots, x_{s-1})$ not lying in the $(s-1)$-space $\Sigma(x_0, x_1, \cdots, x_{s-1})$. Then $\Pi(x_0, x_1, \cdots, x_s) = \Pi(x_0, x_1, \cdots, x_{s-1}) \cap \Pi(x_s)$.

It is sometimes convenient to represent the points by residue classes of integers. A convenient method of doing this is discussed in [1].

Bibliography

1. Gerald Berman, *Finite projective geometries*, Canadian Journal of Mathematics vol. 4 (1952) pp. 302–313.

2. R. C. Bose and K. R. Nair, *Partially balanced incomplete block designs*, Sankhya vol. 4 (1939) pp. 337–372.

3. R. C. Bose and T. Shimamoto, *Classification and analysis of partially balanced incomplete block designs with two associate classes*, Journal of the American Statistical Association vol. 47 (1952) pp. 151–184.

4. W. H. Clatworthy, *A geometrical configuration which is a partially balanced incomplete block design*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 47–55.

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