THE INTEGRAL OF A FUNCTION WITH RESPECT TO A FUNCTION. II

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1. Introduction. If x is a function,[1] y is a function, and G is a real-number set, then by the graph, (x, y, G), of y with respect to x in G we mean the image of G under a transformation T such that if p is in G, then $T(p)$ is the ordered number-pair $x(p)$, $y(p)$.

Consider the following problem in integration. Let U denote the set such that s is a member of U if and only if s is the graph of a function with respect to a function in an interval. We are to select a subset X of U and assign to each member $(v, u, [a, b])$ of X a number $\int_a^b u(x)dv(x)$ so that the following statements are true.

(1.1) If $(v, u, [a, b])$ is in U and $u(x)=0$ for each number x in $[a, b]$, then $(v, u, [a, b])$ is in X and $\int_a^b u(x)dv(x)=0$.

(1.2) Suppose that $(v, u, [a, b])$ is in X, and $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$ is a number-sequence, and $f(x)=a_{11}u(x)+a_{12}v(x)+a_{13}$ and $g(x)=a_{21}u(x)+a_{22}v(x)+a_{23}$ for each number x in $[a, b]$. Then $(g, f, [a, b])$ is in X, and

$$\int_a^b f(x)dg(x) - 2^{-1}[f(a) + f(b)][g(b) - g(a)]$$

$$= (a_{11}a_{22} - a_{12}a_{21})\left\{\int_a^b u(x)dv(x) - 2^{-1}[u(a) + u(b)][v(b) - v(a)]\right\}.$$

(1.3) Suppose that $(v, u, [a, b])$ is in U and that $a<c<b$. Then $(v, u, [a, b])$ is in X if and only if it is true that $(v, u, [a, c])$ and $(v, u, [c, b])$ are in X; moreover, if $(v, u, [a, b])$ is in X, then $\int_a^b u(x)dv(x)=\int_a^c u(x)dv(x)+\int_c^b u(x)dv(x)$.

In passing, we remark that if u is a step-function, v is a step-function, and $[a, b]$ is an interval, then $(v, u, [a, b])$ is in X and the number $\int_a^b u(x)dv(x)$ is specified by (1.1), (1.2), and (1.3).

One solution of this integration problem has been given in [2], [3], and [1], as may be verified from Definition 2.1 and Theorem 2.1 of [1]. Now from Definition 2.1 of [1] it can readily be seen that if u is integrable with respect to v in $[a, b]$, then there is a countable subset G of $[a, b]$ such that the following statement is true: if ϵ is a posi-

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[1] We use the notation and terminology of [1]; in particular, the words "function," "step-function," "interval," "subdivision," "refinement," and "integrable" are used as in [1].

tive number, then there is a subdivision D of $[a, b]$, each of whose terms is in G, such that $|S_E(u, v) - \int_a^b u(x)dv(x)| < \epsilon$ if E is a refinement of D each of whose terms is in G.

This fact suggests the possibility of a new way of defining an integral as a limit of approximating sums (see §2 of this paper) so as to retain the properties (1.1), (1.2), (1.3). Most of the fundamental ideas involved are illustrated by the following example, in which we consider the integral of a totally discontinuous function with respect to a totally discontinuous function.

EXAMPLE 1.1. Suppose that $[a, b]$ is an interval, c is a positive number, $u(x) = c$ and $v(x) = x$ if x is an irrational number, and $c+1 \leqq u(x) \leqq c+10$ and $x+1 \leqq v(x) \leqq x+10$ if x is a rational number. Let G denote the set whose members are a, b, and the irrational numbers between a and b; and let H denote the set whose members are the rational numbers between a and b. Then

(i) the graph (v, u, H) is a singular graph (Definition 2.1),

(ii) G is a summability set (Definition 2.3) for u and v in $[a, b]$,

(iii) $\int_a^b u(x)dv(x) = c(b-a) + 2^{-1}[c + u(b)][v(b) - b] - 2^{-1}[u(a) + c] \cdot [v(a) - a]$, and

(iv) if A is a subdivision of $[a, b]$ and ϵ is a positive number, then there is a refinement B of A such that $|S_C(u, v) - \int_a^b u(x)dv(x)| < \epsilon$ if C is a refinement of B each of whose terms is a term of B or a number in G.

2. **Definitions and lemmas.** We now introduce three definitions upon which the rest of this paper will be based.

DEFINITION 2.1. If u is a function, v is a function, and H is a real-number set, then the statement that (v, u, H) is a singular graph means that if ϵ is a positive number, then there is a countable set I of intervals such that

(i) if $[c, d]$ is in I, then neither c nor d is in H,

(ii) if x is in H, then x is in an interval of I, and

(iii) if $[a_p, b_p]$, $p = 1, 2, 3, \cdots$, are the members of I, and A_p and B_p are subdivisions of $[a_p, b_p]$, then $\sum_{(p)} |S_{A_p}(u, v) - S_{B_p}(u, v)| < \epsilon$.

EXAMPLE 2.1. Suppose that $u(x) = 0$ or 1, according as x is a rational number or an irrational number, and that $v(x) = 1 + x$ or x, according as x is a rational number or an irrational number. Let $[a, b]$ denote an interval such that a and b are irrational numbers, and let H denote the set of all rational numbers between a and b. Then (v, u, H) is a singular graph.

REMARK 2.1. If D and E are subdivisions of $[a, b]$, then $S_D(u, v) + S_D(v, u) = u(b)v(b) - u(a)v(a) = S_E(u, v) + S_E(v, u)$, so that $S_D(v, u)$

$-S_E(v, u) = S_E(u, v) - S_D(u, v)$. Hence if (v, u, H) is a singular graph, then so is (u, v, H).

DEFINITION 2.2. The statement that x is an exceptional number for the functions u and v in the interval $[a, b]$ means that there is a subinterval $[c, d]$ of $[a, b]$ containing x such that if $[p, q]$ is a subinterval of $[c, d]$ then

(i) u is integrable with respect to v in $[p, q]$ if x is not in $[p, q]$,

(ii) u is not integrable with respect to v in $[p, q]$ if x is in $[p, q]$.

EXAMPLE 2.2. Suppose that u is a function such that if x is a real number other than 1, then the limits $u(x+)$ and $u(x-)$ exist, but the limit $u(1+)$ does not exist. Suppose that v is a function of bounded variation such that $v(1+) \neq v(1)$. Then 1 is an exceptional number for u and v in $[0, 2]$.

DEFINITION 2.3. The statement that G is a summability set for the functions u and v in the interval $[a, b]$ means that

(i) G is a subset of $[a, b]$, and a and b are in G, and none of the numbers in G is an exceptional number for u and v in $[a, b]$,

(ii) if ϵ is a positive number, then there is a subdivision D of $[a, b]$, each of whose terms is in G, such that if E is a refinement of G each of whose terms is in G, then $|S_D(u, v) - S_E(u, v)| < \epsilon$, and

(iii) if G is not $[a, b]$ and H is the complement of G in $[a, b]$, then (v, u, H) is a singular graph.

EXAMPLE 2.3. Let G denote the complement in $[a, b]$ of the set H defined in Example 2.1, and let u and v be defined as in Example 2.1. Then G is a summability set for u and v in $[a, b]$; moreover, if ϵ is a positive number, then there is a subdivision D of $[a, b]$, each of whose terms is in G, such that if E is a refinement of D each of whose terms is in G, then $|S_E(u, v) - (b-a)| < \epsilon$.

LEMMA 2.1. *If G is a summability set for u and v in $[a, b]$, then there is just one number k such that the following statement is true*:

(2.1) *If ϵ is a positive number, then there is a subdivision D of $[a, b]$, each of whose terms is in G, such that if E is a refinement of D each of whose terms is in G then $|S_E(u, v) - k| < \epsilon$.*

PROOF. A. Let D_1 denote a subdivision of $[a, b]$, each of whose terms is in G, such that $|S_{D_1}(u, v) - S_E(u, v)| < 1/2$ if E is a refinement of D_1 each of whose terms is in G. For each integer n greater than 1, let D_n denote a refinement of D_{n-1}, each of whose terms is in G, such that $|S_{D_n}(u, v) - S_E(u, v)| < 1/2^n$ if E is a refinement of D_n each of whose terms is in G. Now if m is a positive integer and n is an integer greater than m, then $|S_{D_m}(u, v) - S_{D_n}(u,v)| < 1/2^m$; by Cauchy's convergence criterion, there is a number k such that if n

is a positive integer then $|S_{D_n}(u, v)-k| \leqq 1/2^n$ and therefore $|S_E(u, v)-k| < 1/2^{n-1}$ if E is a refinement of D_n each of whose terms is in G. Hence there is a number k such that (2.1) is true.

B. For $i=1, 2$, suppose that k_i is a number such that if k is k_i then (2.1) is true. Let ϵ denote a positive number, and for $i=1, 2$, let A_i denote a subdivision of $[a, b]$, each of whose terms is in G, such that if E is a refinement of A_i each of whose terms is in G then $|S_E(u, v) -k_i| < \epsilon$. Let E denote the refinement of A_1 whose terms are the terms of A_1 and A_2. Then $|S_E(u, v)-k_1| < \epsilon$, and $|S_E(u, v)-k_2| < \epsilon$, and therefore $|k_1-k_2| < 2\epsilon$ if ϵ is a positive number. Hence $k_1=k_2$. This completes the proof.

LEMMA 2.2. *Suppose that* $[a, b]$ *is an interval, u is a function, v is a function,* ϵ *is a positive number, and* $\{[c_p, d_p]\}_{p=1}^n$ *is a finite sequence of intervals such that*

(i) *if x is in* $[a, b]$ *then x is in one of the intervals* $[c_p, d_p]$, *and*

(ii) *if* C_p *and* D_p *are subdivisions of* $[c_p, d_p]$, $p=1, 2, \cdots, n$, *then* $\sum_{p=1}^n |S_{C_p}(u, v)-S_{D_p}(u, v)| < \epsilon$.

If A is a subdivision of $[a, b]$ *among whose terms are the numbers* c_p *and* d_p *(if any) which are in* $[a, b]$, *and B is a refinement of A, then* $|S_A(u, v)-S_B(u, v)| < \epsilon$.

PROOF. Case I; $n=1$. In this case, $[a, b]$ is a subinterval of $[c_1, d_1]$. Let A denote a subdivision of $[a, b]$, and let B denote a refinement of A. Let C_1 and D_1 denote the subdivisions of $[c_1, d_1]$ whose terms are c_1, d_1, and the terms of A and B, respectively. Then $S_A(u, v)-S_B(u, v) =S_{C_1}(u, v)-S_{D_1}(u, v)$, and hence $|S_A(u, v)-S_B(u, v)| < \epsilon$.

Case II; $n>1$. Let K denote the number sequence $c_1, d_1, c_2, d_2, \cdots, c_n, d_n$. Let A denote a subdivision of $[a, b]$ among whose terms are the terms (if any) of K which are in $[a, b]$, and let B denote a refinement of A. Let C_1 denote the subdivision of $[c_1, d_1]$ whose terms are the terms of A and K which are in $[c_1, d_1]$, and let D_1 denote the subdivision of $[c_1, d_1]$ whose terms are the terms of B and K which are in $[c_1, d_1]$. For $p=2, 3, \cdots, n$, let C_p denote the subdivision of $[c_p, d_p]$ whose terms are the terms of K which are in $[c_p, d_p]$ and the terms (if any) of A which are in $[c_p, d_p]$ but are not terms of $K, C_1, C_2, \cdots, C_{p-1}$; and let D_p denote the subdivision of $[c_p, d_p]$ whose terms are the terms of K which are in $[c_p, d_p]$ and the terms (if any) of B which are in $[c_p, d_p]$ but are not terms of $K, D_1, D_2, \cdots, D_{p-1}$. Then $S_A(u, v)-S_B(u, v)=\sum_{p=1}^n [S_{C_p}(u, v) -S_{D_p}(u, v)]$; so $|S_A(u, v)-S_B(u, v)| \leqq \sum_{p=1}^n |S_{C_p}(u, v)-S_{D_p}(u, v)| < \epsilon$. This completes the proof.

LEMMA 2.3. *Suppose that* G_1 *and* G_2 *are summability sets for u and v* *in* $[a, b]$. *If there is a subinterval* $[c, d]$ *of* $[a, b]$ *such that none of the numbers between* c *and* d *is in* G_1 *and in* G_2, *then* u *is integrable with respect to* v *in* $[c, d]$.

PROOF. Case I; none of the numbers between c and d is in G_2. Since G_2 is a summability set for u and v in $[a, b]$, it follows that if $\epsilon > 0$ then there is an interval $[c_1, d_1]$, of which $[c, d]$ is a subinterval, such that c_1 and d_1 are in G_2 and $|S_C(u, v) - S_D(u, v)| < \epsilon$ if C and D are subdivisions of $[c_1, d_1]$. By Lemma 2.2, if A is a subdivision of $[c, d]$ and B is a refinement of A, then $|S_A(u, v) - S_B(u, v)| < \epsilon$. Hence u is integrable with respect to v in $[c, d]$.

Case II; none of the numbers between c and d is in G_1. For this case, the argument is similar to that used in Case I.

Case III; there are between c and d a number which is in G_1 and a number which is in G_2. Let ϵ denote a positive number. For $i = 1, 2$, let I_i denote a countable set (see Definition 2.1) of subintervals of $[a, b]$ such that

(1) if $[r, s]$ is in I_i, then r and s are in G_i,

(2) if x is between a and b but not in G_i, then there is an interval $[r, s]$ in I_i such that $r < x < s$, and

(3) if the members of I_i are $[a_p, b_p]$, and A_p and B_p are subdivisions of $[a_p, b_p]$, $p = 1, 2, \cdots$, then $\sum_{(p)} |S_{A_p}(u, v) - S_{B_p}(u, v)| < \epsilon/2$.

Now let $[h, k]$ denote a subinterval of $[c, d]$ such that if x is in $[h, k]$ then x is not in G_1 or x is not in G_2. Then $[h, k]$ can be covered by a set s_0 of segments (a_p, b_p) such that the intervals $[a_p, b_p]$ are from I_1 and I_2. Let $[c_p, d_p]$, $p = 1, 2, \cdots, n$, denote a finite subset of s_0 such that the segments (c_p, d_p) cover $[h, k]$. Let D denote the subdivision of $[h, k]$ whose terms are the numbers h and k and the numbers c_p and d_p which are in $[h, k]$. By Lemma 2.2, if E is a refinement of D, then $|S_D(u, v) - S_E(u, v)| < \epsilon$; so u is integrable with respect to v in $[h, k]$. Now either u is integrable with respect to v in $[c, k]$ or c is an exceptional number for u and v in $[a, b]$ and is not in G_1 or G_2, in which case u is integrable with respect to v in $[c, k]$ by the above argument; similarly, if $k < d$, then u is integrable with respect to v in $[k, d]$ and consequently in $[c, d]$. This completes the proof.

LEMMA 2.4. *Suppose that* G_1 *and* G_2 *are summability sets for* u *and* v *in* $[a, b]$ *and that* x *is a number between* a *and* b *which is in* G_2 *but not in* G_1. *If* δ *is a positive number, then there is a subinterval* $[c, d]$ *of* $[a, b]$ *such that*

(i) *each of* c *and* d *is in* G_1 *and in* G_2,

(ii) $c < x < d$, *and*

(iii) *there is a subdivision D of $[c, d]$ such that if E is a refinement of D then* $|S_D(u, v)-S_E(u, v)|<\delta$.

Proof. Case I; G_2 is $[a, b]$. Let H denote the complement of G_1 in $[a, b]$; then x is in H, and (v, u, H) is a singular graph. Hence if δ is a positive number, then there is a subinterval $[c, d]$ of $[a, b]$ such that c and d are in G_1 and in G_2, x is between c and d, and $|S_D(u,v)-S_E(u, v)|<\delta$ if D and E are subdivisions of $[c, d]$.

Case II; G_2 is a proper subset of $[a, b]$. Suppose that δ is a positive number and that $\epsilon=\delta/2$. Let sets I_1 and I_2 be selected as in the proof of Lemma 2.3. Let $[c_1, d_1]$ denote an interval from I_1 such that $c_1<x<d_1$. If there is a number y in $[c_1, x]$ which is in G_1 and in G_2, let c denote such a number y, and let D_1 denote a subdivision of $[c, x]$; if E is a refinement of D_1, then $|S_{D_1}(u, v)-S_E(u, v)|<\epsilon$. If none of the numbers in $[c_1, x]$ is in G_1 and in G_2, let c' denote the smallest number t such that if y is a number in $[a, x]$ which is in G_1 and in G_2 then $y\leqq t$. If c' is in G_1 and in G_2, let c denote c'; by Lemma 2.3, u is integrable with respect to v in $[c, x]$, and hence there is a subdivision D_1 of $[c, x]$ such that $|S_{D_1}(u, v)-S_E(u, v)|<\epsilon$ if E is a refinement of D_1. If c' is not in G_1 and in G_2, let $[c'', d'']$ denote an interval from I_1 or I_2 such that $c''<c'<d''$, and let c denote a number in $[c'', c']$ which is in G_1 and in G_2; by Lemma 2.3, u is integrable with respect to v in $[c', x]$, and by Lemma 2.2 if A is a subdivision of $[c, c']$ and B is a refinement of A then $|S_A(u, v)-S_B(u, v)|<\epsilon/2$; hence there is a subdivision D_1 of $[c, x]$ such that $|S_{D_1}(u, v)-S_E(u, v)|<\epsilon$ if E is a refinement of D_1.

Similarly we find a number d in $[x, b]$ which is in G_1 and in G_2, and a subdivision D_2 of $[x, d]$ such that $|S_{D_2}(u, v)-S_E(u, v)|<\epsilon$ if E if a refinement of D_2. Let D denote the subdivision of $[c, d]$ whose terms are the terms of D_1 and D_2; if E is a refinement of D, then $|S_D(u, v)-S_E(u, v)|<2\epsilon=\delta$. This completes the proof.

Lemma 2.5. *Suppose that for $i=1, 2$, G_i is a summability set for u and v in $[a, b]$ and k_i is a number such that if G is G_i and k is k_i then* (2.1) *is true. Then $k_1=k_2$.*

Proof. To prove the lemma, we show that if $\epsilon>0$ then $|k_1-k_2|<6\epsilon$. Suppose that $\epsilon>0$; for $i=1, 2$, let A_i denote a subdivision of $[a, b]$, each of whose terms is in G_i, such that if E_i is a refinement of A_i each of whose terms is in G_i then $|S_E({}_iu, v)-k_i|<\epsilon$ and $|S_{A_i}(u, v)-S_{E_i}(u, v)|<\epsilon$. Let A denote the subdivision of $[a, b]$ whose terms are the terms of A_1 and A_2.

Case I; each of the terms of A is in G_1 and in G_2. In this case, $|S_A(u, v)-k_1|<\epsilon$ and $|S_A(u, v)-k_2|<\epsilon$; so $|k_1-k_2|<2\epsilon<6\epsilon$.

Case II; there is a term of A which is not in G_1 or not in G_2. Let

$x_1, x_2, \cdots, x_n$ denote the terms of A which are not in G_1 or not in G_2. For $p=1, 2, \cdots, n$, let $[c_p, d_p]$ denote a subinterval of $[a, b]$ and C_p a subdivision of $[c_p, d_p]$ such that

(i) each of c_p and d_p is in G_1 and in G_2,

(ii) $c_p < x_p < d_p$, and

(iii) if D_p is a refinement of C_p then $|S_{C_p}(u, v) - S_{D_p}(u, v)| < \epsilon/n$. Let D denote the subdivision of $[a, b]$ whose terms are the terms of $C_1, C_2, \cdots, C_n$, and A. If each of the terms of D is in G_1, then $|S_D(u, v) - k_1| < \epsilon < 3\epsilon$. Suppose that there is a term of D which is not in G_1; let $t_1, t_2, \cdots, t_m$ denote the terms of D which are not in G_1. If k is one of the first m positive integers, then there is an integer p such that $c_p < t_k < d_p$, and (by Definition 2.1 and Lemma 2.2) there is a subinterval $[r_k, s_k]$ of $[c_p, d_p]$ such that r_k and s_k are in G_1, and $r_k < t_k < s_k$, and $|S_K(u, v) - S_L(u, v)| < \epsilon/2m$ if K and L are subdivisions of $[r_k, s_k]$. Let E denote the refinement of D whose terms are the terms of D and the numbers r_k and s_k, $k=1, 2, \cdots, m$; then $|S_D(u, v) - S_E(u, v)| < \epsilon$. Let F denote the subdivision of $[a, b]$ whose terms are the terms of E which are in G_1; then $|S_E(u, v) - S_F(u, v)| < \epsilon$. Moreover, F is a refinement of A_1 each of whose terms is in G_1; so $|S_F(u, v) - k_1| < \epsilon$. Hence $|S_D(u, v) - k_1| < 3\epsilon$. By a similar argument, $|S_D(u, v) - k_2| < 3\epsilon$; so $|k_1 - k_2| < 6\epsilon$. This completes the proof.

REMARK 2.2. If u is integrable with respect to v in $[a, b]$, then $[a, b]$ is a summability set for u and v in $[a, b]$, and $\int_a^b u(x)dv(x)$ is the number k of Lemma 2.1.

DEFINITION 2.4. Suppose that u is a function, v is a function, and $[a, b]$ is an interval.

(i) The statement that u is summable with respect to v in $[a, b]$ means that there is a summability set for u and v in $[a, b]$.

(ii) if u is summable with respect to v in $[a, b]$, then $\int_a^b u(x)dv(x)$, the integral of u with respect to v in $[a, b]$, is the number k of Lemma 2.1, and $\int_b^a u(x)dv(x) = -\int_a^b u(x)dv(x)$.

3. **Some properties of the integral.** We shall now show that the integral defined in Definition 2.4 has the properties specified in §1. It follows directly from Definition 2.4 that the statement (1.1) is true if X denotes the subset of U such that a member $(v, u, [a, b])$ of U is in X if and only if u is summable with respect to v in $[a, b]$.

THEOREM 3.1. *Suppose that the function u is summable with respect to the function v in the interval $[a, b]$, and $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$ is a number sequence, and $f(x) = a_{11}u(x) + a_{12}v(x) + a_{13}$ and $g(x) = a_{21}u(x) + a_{22}v(x) + a_{23}$ for each number x in $[a, b]$. Then f is summable with respect to g in $[a, b]$, and*

$$\int_a^b f(x)dg(x) - 2^{-1}[f(a) + f(b)][g(b) - g(a)]$$
$$= (a_{11}a_{22} - a_{12}a_{21})\left\{\int_a^b u(x)dv(x) - 2^{-1}[u(a) + u(b)][v(b) - v(a)]\right\}.$$

PROOF. If D is a subdivision of a subinterval $[c, d]$ of $[a, b]$, then

$$\begin{aligned} S_D(f, g) &= a_{21}S_D(f, u) + a_{22}S_D(f, v) \\ &= a_{11}a_{21}S_D(u, u) + a_{12}a_{21}S_D(v, u) + a_{13}a_{21}S_D(1, u) \\ &\quad + a_{11}a_{22}S_D(u, v) + a_{12}a_{22}S_D(v, v) + a_{13}a_{22}S_D(1, v) \\ &= 2^{-1}a_{11}a_{21}u^2\Big|_c^d + a_{12}a_{21}uv\Big|_c^d - a_{12}a_{21}S_D(u, v) + a_{13}a_{21}u\Big|_c^d \\ &\quad + a_{11}a_{22}S_D(u, v) + 2^{-1}a_{12}a_{22}v^2\Big|_c^d + a_{13}a_{22}v\Big|_c^d . \end{aligned}$$

Hence if D and E are subdivisions of $[c, d]$, then

$$S_D(f, g) - S_E(f, g) = (a_{11}a_{22} - a_{12}a_{21})[S_D(u, v) - S_E(u, v)].$$

Hence if G is a summability set for u and v in $[a, b]$, then G is a summability set for f and g in $[a, b]$. Moreover, if D is a subdivision of $[a, b]$ and E is the subdivision of $[a, b]$ whose only terms are a and b, then

$$S_D(f, g) - 2^{-1}[f(a) + f(b)][g(b) - g(a)]$$
$$= (a_{11}a_{22} - a_{12}a_{21})\{S_D(u, v) - 2^{-1}[u(a) + u(b)][v(b) - v(a)]\}.$$

The theorem now follows at once.

COROLLARY 3.1a. *If u is summable with respect to v in $[a, b]$, then v is summable with respect to u in $[a, b]$, and $\int_a^b u(x)dv(x) = u(b)v(b) - u(a)v(a) - \int_a^b v(x)du(x)$.*

LEMMA 3.2a. *If the function u is summable with respect to the function v in the interval $[a, b]$ and x is a number in $[a, b]$, then x is not an exceptional number for u and v in $[a, b]$.*

PROOF. Let G denote a summability set for u and v in $[a, b]$. If x is in G, then by Definition 2.3, x is not an exceptional number for u and v in $[a, b]$. Suppose that x is not in G; let H denote the complement of G in $[a, b]$; then x is in H, and (v, u, H) is a singular graph. Let ϵ denote a positive number. Then there is a subinterval $[c_1, d_1]$ of $[a, b]$ such that c_1 and d_1 are in G, x is between c_1 and d_1, and

$|S_{C_1}(u, v)-S_{D_1}(u, v)|<\epsilon/2$ if C_1 and D_1 are subdivisions of $[c_1, d_1]$. Suppose that there is a subinterval $[c, d]$ of $[a, b]$ containing x such that if $[p, q]$ is a subinterval of $[c, d]$ which does not contain x then u is integrable with respect to v in $[p, q]$. If $c\leqq x<d$, let p denote a number less than d between x and d_1, and let C_2 denote a subdivision of $[p, d]$ such that $|S_{C_2}(u, v)-S_{D_2}(u, v)|<\epsilon/2$ if D_2 is a refinement of C_2, and such that d_1 is a term of C_2 if $d_1<d$. By Lemma 2.2, there is a subdivision C_3 of $[x, p]$ such that if D_3 is a refinement of C_3 then $|S_{C_3}(u, v)-S_{D_3}(u, v)|<\epsilon/2$; hence it follows that there is a subdivision C of $[x, d]$ such that if D is a refinement of C then $|S_C(u, v)-S_D(u, v)|<\epsilon$; so u is integrable with respect to v in $[x, d]$. Similarly, if $c<x\leqq d$, then u is integrable with respect to v in $[c, x]$. Hence u is integrable with respect to v in $[c, d]$ and consequently in each subinterval of $[c, d]$; so x is not an exceptional number for u and v in $[a, b]$. This completes the proof.

THEOREM 3.2. *Suppose that u is summable with respect to v in $[a, b]$, and that $a<c<b$. Then u is summable with respect to v in $[a, c]$ and in $[c, b]$; and $\int_a^c u(x)dv(x)+\int_c^b u(x)dv(x)=\int_a^b u(x)dv(x)$.*

PROOF. Let G denote a summability set for u and v in $[a, b]$, let G_1 denote the set whose members are c and the members of G which are in $[a, c]$, and let G_2 denote the set whose members are c and the members of G which are in $[c, b]$. If c is in G, it readily follows from Definition 2.3 and Lemma 2.2 that G_1 and G_2 are summability sets for u and v in $[a, c]$ and $[c, b]$, respectively, and consequently that the conclusion of the theorem is true. Suppose that c is not in G. By Lemma 3.2a, c is not an exceptional number for u and v in $[a, b]$ and hence is not an exceptional number for u and v in $[a, c]$ or in $[c, b]$. Since c is not in G, it follows that if ϵ is a positive number, then there is a subinterval $[p, q]$ of $[a, b]$ such that p and q are in G, c is between p and q, and $|S_P(u, v)-S_Q(u, v)|<\epsilon$ if P and Q are subdivisions of $[p, q]$. From Lemma 2.2 and Definition 2.3 it readily follows that G_1 and G_2 are summability sets for u and v in $[a, c]$ and $[c, b]$, respectively, and consequently that the conclusion of the theorem is true. This completes the proof.

THEOREM 3.3. *Suppose that $[a, b]$ is an interval, H is a subset of the segment (a, b), G is the complement of H in $[a, b]$, and u, v, u_1, v_1 are functions such that*

(i) *u_1 is integrable with respect to v_1 in $[a, b]$,*

(ii) *each of (v, u, H) and (v_1, u_1, H) is a singular graph,*

(iii) *$u(x)=u_1(x)$ and $v(x)=v_1(x)$ if x is in G, and*

(iv) *if x is in G then x is not an exceptional number for u and v in* $[a, b]$.
Then G is a summability set for u and v in $[a, b]$, and $\int_a^b u(x)dv(x) = \int_a^b u_1(x)dv_1(x)$.

PROOF. By hypothesis u_1 is integrable with respect to v_1 in $[a, b]$; so if x is in G, then x is not an exceptional number for u_1 and v_1 in $[a, b]$, and since (v_1, u_1, H) is a singular graph it can readily be seen that G is a summability set for u_1 and v_1 in $[a, b]$. But if D is a subdivision of $[a, b]$ each of whose terms is in G, then $S_D(u, v) = S_D(u_1, v_1)$, and consequently G is a summability set for u and v in $[a, b]$ and $\int_a^b u(x)dv(x) = \int_a^b u_1(x)dv_1(x)$. This completes the proof.

REFERENCES

1. R. E. Lane, *The integral of a function with respect to a function*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 59–66.

2. H. L. Smith, *On the existence of the Stieltjes integral*, Trans. Amer. Math. Soc. vol. 27 (1925) pp. 491–515.

3. H. S. Kaltenborn, *Existence conditions and a substitution theorem for Stieltjes mean integrals*, Tôhoku Math. J. vol. 44 (1938) pp. 1–11.

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