SYLOW \( p \)-SUBGROUPS OF THE GENERAL LINEAR GROUP
OVER FINITE FIELDS OF CHARACTERISTIC \( p \)

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If \( K \) is the finite field \( GF(q) \) with \( q = p^k \) elements then the general linear group \( GL_n(K) \) has order

\[ q^{n(n-1)/2} (q - 1) \cdots (q^n - 1). \]

Let \( e_{ij} \) denote the matrix with the 1 of \( K \) in the \((i, j)\) position and 0 elsewhere; we shall call any matrix of the form \( 1 + \sum_{i<j} a_{ij} e_{ij} \) \( 1 \)-triangular. The group \( G_n \) of all \( 1 \)-triangular matrices in \( GL_n(K) \) is a Sylow \( p \)-subgroup of \( GL_n(K) \). We shall often write \( G \) for \( G_n \) if this is unambiguous. \( p \) is assumed throughout to be an odd prime.

The generators \( 1 + a e_{i,i+1} \) and the fundamental relations connecting them are studied carefully in a recent paper by Pavlov\(^2\) (for the particular case \( q = p \)) and we have therefore mentioned them briefly in the opening paragraph.

When \( i < j \) the group \( P_{ij} \) of all \( 1 + a e_{ij} (a \in K) \) is isomorphic to the additive group of \( K \). Any subgroup \( P \) of \( G \) generated by these \( P_{ij} \) is characterised by a partition diagram \([P]\). These partition diagrams bear a strong resemblance to the row of "hauteurs" which define the "sous-groupes parallélotopiques" of the Sylow \( p \)-subgroups of the symmetric groups on \( p^n \) symbols, studied by Kaloujnine.\(^3\) A necessary and sufficient condition is given for the partition subgroup \( P \) to be normal in \( G \) and if \( P' = (P, G), P^*/P = \text{centre of } G/P \), the duality between \( P' \) and \( P^* \) is emphasised by constructing their partition diagrams.

Certain "diagonal" automorphisms are introduced and used to prove that any characteristic subgroup of \( G \) is a normal partition subgroup. The maximal abelian normal subgroups are fully investigated and used in conjunction with the symmetry about the second diagonal to give a simple combinatorial proof that the characteristic subgroups of \( G \) are precisely those given by symmetric normal partitions. In the last section we finally identify the group of automorphisms of \( G \).

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\(^1\) Dickson, Linear groups.


\(^3\) L. Kaloujnine, La structure des \( p \)-groupes de Sylow des groupes symétriques finis, Ann. École Norm. vol. 65 (1948) pp. 239–276.
This paper is substantially the content of Chapter 5 of my Cambridge University Ph.D. Thesis (1953) and I should like to acknowledge here my gratitude to Professor Philip Hall who supervised this research in such a kind and encouraging way.

1. Generators of \( G_n \).

\[
e_{ij}e_{hk} = \begin{cases} e_{ik} & \text{if } j = h, \\ 0 & \text{if } j \neq h. \end{cases}
\]

If \( A = 1 + \sum_{i<j} a_{ij}e_{ij} \), then \( r_s = \prod_{j>s} (1 + a_{s}e_{s}) = 1 + \sum_{j>s} a_{s}e_{sj} \) has the same \( s \)th row as \( A \). Then \( r_{n-1}r_{n-2} \cdots r_1 = A \). Thus the set of all \( 1 + ae_{ij} \) \((a \in K, i < j)\) generate \( G \).

Further if \( u < v < w, a, b \in K \), we have the fundamental commutator relation

\[
(1) \quad (1 + ae_{uv}, 1 + be_{vw}) = 1 + abe_{uw}.
\]

Putting \( b = 1; u = i, v = i + 1 \) and \( w = i + 2, i + 3, \cdots \) in succession we see that the set of elements \( 1 + ae_{i,i+1} \) \((a \in K; i = 1, \cdots, n - 1)\) generate the group \( G \).

2. The lower central series of \( G_n \). We define \( H_k \) to be the set of all \( A \) for which \( a_{ij} = 0 \) for \( 0 < j - i < k \). If we write \( 0 \to 0_1 \to \cdots \) for the derived series of \( G \) we have the following

**Theorem 1.** (i) The lower central series of \( G_n \) coincides with the series

\[
H_1 > H_2 > \cdots > H_n = 1.
\]

(ii) \((H_k, H_m) = H_{k+m}\).

(iii) \( \theta_k = H_{2k} \).

**Proof.** Let \( V_k \) be the set of all \( L \) for which \( 1 + L \in H_k \). We verify immediately that \( V_k V_m \subset V_{k+m} \). It follows that \( H_k \) is a group. Moreover if \( 1 + L \in G \), then \( 1 - L + L^2 \cdots \) terminates and must therefore be \((1 + L)^{-1}\).

Say \( A = 1 + L \in H_k \) and \( B = 1 + M \in H_m \) then

\[
(A, B) = (1 + L)^{-1}(1 + M)^{-1} + L - ML + O_{2m+k} \{1 + M\}
= (1 + L)^{-1}\{1 + L + LM - ML + O_{2m+k}\}
= 1 + LM - ML + O_{2m+k} + O_{2k+m}
= 1 + O_{k+m},
\]

where \( O_k \) denotes "some element of \( V_k \)."

In other word \((H_k, H_m) \subset H_{k+m}\).

\( H_k \) is generated (with some generators to spare, in general) by the set of all \( 1 + a_{ij}e_{ij} \) \((a_{ij} \in K, j - i \geq k)\). If now \( w - u \geq k + m \) we may find \( v \) so that \( v - u \geq k \) and \( w - v \geq m \), and we obtain the generators of
\[ H_{k+m} \text{ in the form } 1 + ae_{uv} = (1 + ae_{uw}, 1 + e_{vw}). \] Hence \((H_k, H_m) \supset H_{k+m}\), and so finally \((H_k, H_m) = H_{k+m}\).

In particular \((H_m, H_1) = H_{m+1}\). Since \(H_1 = G\), \(H_1 > H_2 > \cdots > H_n = 1\) is the lower central series of \(G_n\).

The third part of the theorem follows immediately from the second by induction.

3. The partition subgroups. If \(i < j\) the group \(P_{ij}\) of all \(1 + ae_{ij}(a \in K)\) is isomorphic to the additive group of \(K\) and so is elementary abelian of order \(q\). Any subgroup \(P\) of \(G\) generated by a selection of these \(P_{ij}\) is called a partition subgroup. Such a subgroup may be characterised by a "partition" diagram \(|P|\) in the natural way. For example (if \(n \geq 4\)) the group generated by \(P_{12}\) and \(P_{24}\) contains also the subgroup \(P_{14}\) and \(|P|\) consists of the squares \((1, 2), (2, 4), (1, 4)\). The sequence of diagrams for the lower central series is obtained from the whole diagram (representing \(G\)) by removing successive diagonals \(j-i = 1, 2, \ldots\).

Theorem 2. A necessary and sufficient condition for the partition subgroup \(P\) to be normal in \(G\) is that the boundary of \(|P|\) should move monotonically downward and to the right.

Proof. If \(N\) is the least normal subgroup containing \(1 + ae_{ij}\), by the identity (1) it is clear that \(N\) must also contain \(P_{ui}\) and \(P_{iv}\) where \(u < i\) and \(v > j\). Further since \(P_{iv} \subseteq N\) we have \(P_{uw} \subseteq N\) where \(u < i, v > j\). If \(|N_{ij}|\) consists of the squares \((u, v)\) with \(u \leq i, v \geq j\) and if \(|N_{ij}'|\) is \(|N_{ij}|\) omitting \((i, j)\), then \(N\) must contain \(N_{ij}'\). The least normal subgroup containing \(P_{ij}\) is \(N_{ij}\). We shall find it convenient to refer to this process as "completing the rectangle." Now if \(P\) is any normal partition subgroup and if \((i, j)\) is any square in \(|P|\), then \(P\) must contain \(N_{ij}\). Conversely, the product of several \(N_{ij}\) is a normal subgroup of \(G\). These remarks are equivalent to the statement of the theorem.

Given two distinct squares \((i, j), (u, v)\) in \(|G|\); if \(u \leq i, v \geq j\) we shall say \((i, j)\) covers \((u, v)\). When \(|P|\) is a normal partition we shall say \(|P|\) covers \((u, v)\) if some square of \(|P|\) covers \((u, v)\). If \((u, v)\) covers some square outside \(|P|\) we shall say \((u, v)\) avoids \(|P|\).

When \(P\) is a normal partition subgroup we may define the groups \(P' = (P, G)\) and \(P^*\) where \(P^*/P = \text{centre of } G/P\). Then \(P'\) and \(P^*\) are again normal partition subgroups. More precisely

Theorem 3. \(|P'|\) consists of the squares covered by \(|P|\), and \(|P^*|\) consists of the squares which do not avoid \(|P|\).

Proof. Let \(|N|\) be the set of squares covered by \(|P|\). By the proc-
ess of completing the rectangle we see that $P'$ contains $N$.

Now $N$ is the product of normal subgroups $N_i'$ and so is normal. If $(i, j) \in |P|$, then $(1 + ae_{ij}, 1 + be_{km}) \in N$. Any commutator $(z, t)$ where $z \in P$, $t \in G$ may be expanded by application of the rule $(xy, rs) = (x, s)^y(x, r)^z(y, r)^z(y, s)$. Hence $(P, G) \subseteq N$.

Let $|P|$ be the set of squares which do not avoid $|P|$. We obtain $|P|$ by adding one square to each row of $|P|$ except when this new square covers a square outside $|P|$. Clearly $(P, G) \subseteq P$. If $A = 1 + \sum_{i<j} a_{ij}e_{ij} \in P$ then $a_{ij} \neq 0$ for some $(i, j)$ avoiding $|P|$ and $(A, G) \notin P$. Hence $P = P^*$. [We notice that the notation $N'_i$ already used is consistent with that of Theorem 3.]

Theorem 3 shows how strong is the duality between the groups $P'$ and $P^*$. In particular we have as an immediate corollary

THEOREM 4. The upper and lower central series of $G$ coincide.

4. The diagonal automorphisms. If $W$ is the diagonal matrix

$$
\begin{pmatrix}
w_1 & & \\
w_2 & & \\
& \ddots & \\
w_n & & 
\end{pmatrix}
$$

in $GL_n(K)$ and $A = 1 + \sum_{i<j} a_{ij}e_{ij} \in G_n$, then $W^{-1}AW = 1 + \sum_{i<j} a_{ij}^*e_{ij}$ where $a_{ij}^* = w_i^{-1}a_{ij}w_i$. Let $D$ be the group of all such $W$.

PROPOSITION. $DG_n$ is the normalizer of $G_n$ in $GL_n(K)$.

PROOF. Clearly $DG_n$ is contained in this normalizer.

Suppose $M = \sum_{i,j} b_{ij}e_{ij}$ where $b_{uv} \neq 0$ $(u > v)$, and $v$ is as small as possible with respect to this property.

On the one hand $(1 + e_{vu})M = M + \sum_j b_{uj}e_{uj}$ and this differs from $M$ in the $(v, v)$ position. On the other hand $M(1 + \sum r < s a_{rs}e_{rs})$ has in the $(v, v)$ position the element $b_{vv} + \sum b_{uv} \sum r < s a_{rs} = b_{vv}$ since the choice of $b_{uv}$ implies that $b_{uv} = 0$ for all $r < v$. Now $1 + e_{vu} \in G_n$ and we have shown that $M^{-1}(1 + e_{vu})M \notin G_n$. Thus $M$ does not belong to the normalizer of $G_n$ in $GL_n(K)$.

Any automorphism of $G$ of the form $A \rightarrow W^{-1}AW$ where $W \in D$ is called a diagonal automorphism. Let $D$ be the group of all diagonal automorphisms.

5. The normal partition subgroups. It is now possible to prove the following

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Theorem 5. Any subgroup of $G$ which is invariant under the inner and diagonal automorphisms is a normal partition subgroup.

Proof. Any matrix of $G_{n+1}$ is expressible in the form

$$\begin{pmatrix} 1 & a \\ 0 & A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = UV, \text{ say}$$

where $A \in G_n$ and $a$ is a row with elements in $K$.

The group of all $V$ is elementary abelian of order $q^n$ and is normal in $G_{n+1}$. In this way it is possible to express $G_{n+1}$ as the split extension $G_{n+1} \cong G_n H$ ($G_n \cap H = 1$).

The theorem is true for $G_2$ and we assume it to be true for $G_n$. Suppose $R$ is a subgroup of $G_{n+1}$ which is invariant under the inner and diagonal automorphisms of $G_{n+1}$. Then $R \cap G_n$ is a subgroup of $G_n$ which is invariant under the inner and diagonal automorphisms of $G_n$ and so by the induction hypothesis is a normal partition subgroup of $G_n$.

$R \cap H$ is a subgroup of $H$ which is normal in $G_{n+1}$ and invariant under diagonal automorphisms. Hence $H$ is of the form $N_{1j}$. $\{\text{If } a = (\alpha_2, \ldots, \alpha_{n+1}) \text{ and } \alpha_j \neq 0 \text{ then } H \text{ contains } P_{1,n+1}, P_{1n}, \ldots, P_{1j}\}$

It is now sufficient to show that $R = (R \cap G_n)(R \cap H)$ for then the theorem follows by induction.

Clearly $R \supset (R \cap G_n)(R \cap H)$. The most general element of $R$ is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = UV, \text{ say}.$$  

If

$$W = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

then

$$W^{-1} UVW = UV^2 \in R \quad (p \neq 2).$$

Hence $U, V \in R$. In other words $R \subset (R \cap G_n)(R \cap H)$.

Remark. Since the diagonal automorphisms clearly leave invariant any partition subgroup, the converse of Theorem 5 is also true and so we may characterise the normal partition subgroups as those which are left invariant by the inner and diagonal automorphisms.

There is a further important automorphism of $G_n$ which we may regard as a symmetry about the second diagonal:
\[ \tau: 1 + \sum a_{ij}e_{ij} \to 1 + \sum b_{ij}e_{ij} \quad \text{where} \quad b_{ij} = a_{n+1-j,n+1-i}. \]

In view of this we have the important

**Corollary.** Every characteristic subgroup of \( G \) is a "symmetric" normal partition subgroup.

6. The maximal abelian normal subgroups. The derived group \( \theta_1 \) of \( G \) gives the maximal abelian quotient group. A natural dual of \( \theta_1 \) would be a maximal abelian normal subgroup. We shall determine the set of all maximal abelian normal subgroups of \( G \).

If \( A_i = N_{i,i+1} \) (\( i = 1, 2, \cdots, n-1 \)) then \( A_i \) is clearly a normal (partition) subgroup. If \( (u, v) \) and \( (u', v') \in |A_i| \) then \( u' \leq i < v \) and \( (P_{uv}, P_{u'v'}) = 1 \). Hence \( A_i \) is abelian. If \( z \in A_i \), that is if

\[ z = 1 + \sum_{u<v} a_{uv}e_{uv} \]

and some \( a_{uv} \neq 0 \) where \( u > i \) or \( v \leq i \), then \( Gp[z, A_i] \) is not abelian.

(i) Say \( u > i \). \((1-e_{iu})z(1+e_{iu}) \) differs from \( z \) in \((i, v)\) position.

(ii) Say \( v \leq i \). \((1-e_{iv,i+1})z(1+e_{iv,i+1}) \) differs from \( z \) in the \((u, i+1)\) position.

We have now shown that \( A_i \) is a maximal abelian normal subgroup of \( G_n \) (\( i = 1, 2, \cdots, n-1 \)).

A necessary and sufficient condition for \( x \in G \) to belong to a maximal abelian normal subgroup of \( G \) is that \( x \) should commute with all its conjugates in \( G \).

Consider \( G_3 \): if \( x = 1+e_{12}+e_{23} \) then \( x \) commutes with its conjugates all of which have the form \( x+ae_{13} \) and so \( x \) is in a maximal abelian normal subgroup (clearly neither \( A_1 \) nor \( A_2 \)). Though the \( A_i \) are the only partition subgroups which are maximal abelian normal we must expect other types of maximal abelian normal subgroups in general.

Suppose

\[ 1 + L = 1 + \sum_{u<v} a_{uv}e_{uv} \quad (a_{uv} \in K), \]
\[ 1 + M = 1 + \sum_{u<v} b_{uv}e_{uv} \quad (b_{uv} \in K) \]

then \((1+L)(1+M) = 1+L+M+LM \) and so \( 1+L, 1+M \) commute if and only if \( L, M \) commute.

Now

\[ (1-e_{ij})(1+L)(1+e_{ij}) = 1 + L + Le_{ij} - e_{ij}L \]

---

since \( e_{ij}L e_{ij} = 0 \). Suppose \( 1 + L \) belongs to a maximal abelian normal subgroup; then we require \( L \) to commute with \( L e_{ij} - e_{ij}L \), in other words we require \( L e_{ij}L - e_{ij}L^2 - L^2 e_{ij} + L e_{ij}L = 0 \) or \( 2L e_{ij}L = e_{ij}L^2 + L^2 e_{ij} \).

Now

\[
L e_{ij} = \sum_{u < i} a_{ui} e_{uj}, \quad e_{ij}L = \sum_{j < v} a_{jv} e_{iv}.
\]

Hence we require

\[
2 \sum_{j < t} \sum_{u < i} a_{ui} a_{ji} e_{ut} = \sum_{j < v} a_{jv} a_{vi} e_{vt} + \sum_{u < v} a_{vu} a_{ui} e_{wj}.
\]

Each of these three sums belongs to a separate part of the partition diagram of \( G \), and so they all vanish \([p \neq 2]\).

We thus have the following three sets of equations:

(i) \[
a_{1i} a_{j,j+1} = a_{1i} a_{j,j+2} = \cdots = a_{1i} a_{jn} = 0, \\
a_{2i} a_{j,j+1} = a_{2i} a_{j,j+2} = \cdots = a_{2i} a_{jn} = 0, \\
\cdots \cdots \cdots \cdots \cdots, \\
a_{i-2} a_{j,j+1} = a_{i-2} a_{j,j+2} = \cdots = a_{i-2} a_{jn} = 0, \\
a_{i-1} a_{j,j+1} = a_{i-1} a_{j,j+2} = \cdots = a_{i-1} a_{jn} = 0;
\]

(ii) \[
a_{j,j+1} a_{j+1,j+2} = 0, \\
a_{j,j+1} a_{j+1,j+2} + a_{j,j+2} a_{j+2,j+3} = 0, \\
\cdots \cdots \cdots \cdots, \\
a_{j,j+1} a_{j+1,n} + a_{j,j+2} a_{j+2,j+2,n} + \cdots + a_{j,n-1} a_{n-1,n} = 0, \\
\]

(iii) \[
a_{12} a_{2i} + a_{13} a_{3i} + \cdots + a_{1,i-1} a_{i-1,i} = 0, \\
a_{23} a_{3i} + \cdots + a_{2,i-1} a_{i-1,i} = 0, \\
\cdots \cdots \cdots \cdots \cdots, \\
a_{i-3} a_{i-2,i} + a_{i-3} a_{i-1,i} = 0, \\
a_{i-2,i-1} a_{i-1,i} = 0.
\]

If there is one element \( a_{uv} \) in the diagonal \( v - u = 1 \) which does not vanish, then by the last equation of (i) every other element in the same diagonal which is not adjacent to \((u, v)\) must vanish, also the first equation of (ii) or the last equation of (iii) show that the adjacent ones vanish. Hence if \( 1 + \sum_{u < v} a_{uv} e_{uv} \) belongs to a maximal abelian normal subgroup and has one nonzero element in the diagonal \( v - u = 1 \), then all the other elements in this diagonal vanish.

Suppose now that the \( v \)th column is the first which is not composed entirely of zeros and \( a_{uv} \) the last nonvanishing element of it. In other
words \( a_{uv} \neq 0 \) and \( a_{ij} = 0 \) if \( j < v \) and also if \( j = v, i > u \). If \( v = n, 1 + L \in A_{n-1} \) so we assume \( v < n \).

There are two cases to consider: (a) \( u > 1 \), (b) \( u = 1 \).

(a) We may take \( i = v \) in (i) and this gives
\[
a_{uv}a_{j,j+1} = a_{uv}a_{j,j+2} = \cdots = 0
\]
provided \( j > v \). Thus \( a_{rs} = 0 \) whenever \( r > v \).

Take \( j = u \) in (ii), then in the equation
\[
a_{u,u+1}a_{u+1,m} + \cdots + a_{uv}a_{vm} + \cdots + a_{u,m-1}a_{m-1,m} = 0
\]
all the \( a_{rs} \) for which \( s < v \) vanish by our choice of \( a_{uv} \), and all the \( a_{rs} \)
for which \( r > v \) vanish by the result we have just proved. Thus
\( a_{vm} = 0 \) (all \( m > v \)).

Finally we have \( a_{rs} = 0 \) whenever \( r \geq v \), and we now see that in this
case \( 1 + L \in A_{v-1} \).

(b) We may take \( i = v \) in (i) and we find just as before that \( a_{rs} = 0 \)
whenever \( r > v \).

The first equation of (iii) is
\[
a_{12}a_{2i} + a_{13}a_{3i} + \cdots + a_{1v}a_{vi} + \cdots + a_{1,i-1}a_{i-1,i} = 0.
\]
Now \( a_{12} = \cdots = a_{1,v-1} = 0 \) by our choice of \( a_{uv} \), and \( a_{rs} = 0 \) whenever
\( r > v \) by above, so that only one term remains in the equation. Thus
\( a_{vi} = 0 \) (\( v < i < n \)).

Finally \( a_{rs} = 0 \) whenever \( r \geq v \) except possibly \( a_{vn} \). If \( a_{vn} = 0 \), then
just as before \( 1 + L \in A_{v-1} \).

However in fact \( a_{vn} \) need not be zero and each of its possible \( q - 1 \)
nonzero values gives us a new maximal abelian normal subgroup.
Any normal subgroup containing for example \( x = 1 + e_{1v} + ce_{vn} \) must
contain all the conjugates of \( x \) in \( G \). Now \( x^{-1} - 1 - e_{1v} - ce_{vn} + ce_{1n} \) and
\( (1 - ae_{vw})x(1 + ae_{uw}) = x + ae_{1w} \). Hence any normal subgroup contain-
ing \( x \) must contain all \( 1 + ae_{lw} \) for \( w > v, a \in K \) and similarly must con-
tain all \( 1 + ae_{wn} \) for \( w < v, a \in K \).

Suppose now that \( y \) belongs to an abelian normal subgroup contain-
ing \( x \), and say \( y = 1 + \sum i<j a_{ij}e_{ij} \). Then \( y \) must commute with \( x \) and
also with \( 1 + e_{lw} \) (all \( w > v \)) and \( 1 + e_{wn} \) (all \( w < v \)). This shows that
\( a_{rs} = 0 \) if \( r \geq v \), also if \( s \leq v \), except possibly \( a_{1v} \neq 0 \) or \( a_{vn} \neq 0 \) but in
this case \( ca_{1v} = a_{vn} \).

Thus \( N_v(c) = Gp[1 + ae_{1v} + cae_{vn}, (a \in K)] \) \( N_{v-1,v+1} \) is the unique
maximal abelian normal subgroup containing \( x \).

The results of this section may be summarised in

**Theorem 6.** The maximal abelian normal subgroups of \( G_n \) fall into
two distinct classes:

\[
A_i = N_{i,i+1} \quad (i = 1, 2, \cdots, n - 1),
\]
7. The characteristic subgroups. We have the following fundamental

**Theorem 7.** The characteristic subgroups of $G$ are precisely the normal partition subgroups whose partitions are symmetric about the second diagonal.

**Proof.** We consider the effect of an automorphism $\theta$ on the maximal abelian normal subgroups. Certainly it is clear that these must be permuted among themselves. All of the "exceptional" maximal abelian normal subgroups $N_v(c)$ except for $v = 2$ are contained in $H_2$ and $H_2$ is characteristic in $G$. Also no $A_i$ is contained in $H_2$ so we expect the $A_i$ (1 < $i$ < $n$ - 1) to be permuted by $\theta$. These $A_i$ divide naturally into pairs of groups with the same order, and for example we see that $A_2$ transforms under $\theta$ into itself or into $A_{n-2}$. Moreover $\theta$ leaves $A_2$ invariant if and only if $\theta$ leaves $A_{n-2}$ invariant. Hence both $A_2A_{n-2}$ and $N_{2,n-1}=N_2\cap A_{n-2}$ are characteristic subgroups of $G_n$.

The join of $A_1$, $A_{n-1}$ and $N_2(c)$ is just $A_1A_{n-1}$ and this again is characteristic in $G_n$.

If we write $r' = n + 1 - r$, then $\tau$ sends $P_r$ into $P_{r'}$. Any symmetric normal partition subgroup may be built up as a join of $N_{r_1}N_{s_1}$. ($r = 1, 2, \cdots$). But these may all be obtained as intersections of groups which we have shown to be characteristic. For example we intersect $A_2A_{n-2}$ successively with $A_3A_{n-3}$, $A_4A_{n-4}$, $\cdots$ and then the square partition subgroups $N_{rr'}$ to obtain every $N_{2r}N_{s_1}$. Combining these results with the corollary to Theorem 5 we have the above theorem.

8. The automorphisms of $G_n$. Since the automorphisms of $G$ have been completely determined by Palov$^2$ for the case of a ground field with $p$ elements, we shall sketch the parts of this section which are merely generalizations of his work, and we shall also try as far as possible to use his notation.

The group $\Gamma$ of inner automorphisms is isomorphic to $G_n/H_{n-1}$ and so has order $q^{(n^2-n-2)/2}$.

The diagonal automorphism induced by the diagonal matrix $W$ is the identity if and only if $W$ is a scalar matrix. Hence $\mathcal{D}$ has order $(q-1)^{n-1}$.

The ground field $K$ may be regarded as a vector space of dimension $k$ over the field $GF(p)$ of integers mod $p$. Let $a_1, \cdots, a_k$ be a basis. The group $GL_k(p)$ of all nonsingular linear transformations of $K$ induces a group $\mathcal{L}$ of automorphisms of $G$: if $g \in GL_k(p)$ then $\gamma$ is the
induced automorphism which maps the generators \(1+a_{i}e_{r,r+1}\) into \(1+a_{i}e_{r,r+1}\) \((i = 1, \ldots, k; r = 1, \ldots, n-1)\).

\(\mathcal{L} \cap \mathcal{D}\) consists of the automorphisms induced by matrices of \(D\) of the form

\[
\begin{pmatrix}
a \\
a^2 \\
\vdots \\
a^n
\end{pmatrix},
\]

\(a \neq 0, a \in K\).

For each \(i = 1, \ldots, k; r = 1, \ldots, n-1\) there is a central automorphism \(\tau_i\) which maps the one generator \(1+a_{i}e_{r,r+1}\) into \(1+a_{i}e_{r,r+1}+b_{i}e_{n}\) (where \(b_{i}\) is an arbitrary element of \(K\), and leaves the other generators invariant. For \(r = 1\) and \(r = n-1\) these are already inner automorphisms. Let \(Z\) be the group generated by \(\tau_i\) \((i = 1, \ldots, k; r = 2, \ldots, n-2)\), then \(Z\) is elementary abelian of order \(q^{k(n-3)}\).

There are two types of extremal automorphisms

\[
\sigma_1(b) : 1 + a_{e_{12}} \rightarrow 1 + a_{e_{12}} + a_{b}e_{2n} \quad (b \in K),
\]

and

\[
\sigma_2(b) : 1 + a_{e_{n-1,n}} \rightarrow 1 + a_{e_{n-1,n}} + a_{b}e_{n,n-1} \quad (b \in K).
\]

The group \(U\) generated by the extremal automorphisms is elementary abelian of order \(q^2\). We write \(\mathcal{P} = ZU\). (This is a direct product.)

**Theorem 8.** The group \(\mathcal{A}\) of all automorphisms of \(G\) is generated by the subgroups \([r], \mathcal{L}, \mathcal{D}, \mathcal{J}, \mathcal{P}\).

**Proof.** If \(\alpha\) is an automorphism which leaves \(H_2\) elementwise invariant and which induces the identity automorphism on \(G/H_2\), then \(\alpha\) may be obtained by multiplying each element of \(G_n\) by an element in the centre \((H_{n-2})\) of \(H_2\). The central automorphisms are clearly of this type.

If \((1+e_{r,r+1})^\alpha = 1+e_{r,r+1}+b_{e_{2n}}\) and \(r > 1\), by commuting with \(1+e_{12}\) we find an element in \(H_2\) which is not invariant unless \(b = 0\).

If \((1+e_{12})^\alpha = 1+e_{12}+b_{e_{2n}}\), then since \(1+e_{12}, 1+ae_{12}\) commute we must have \((1+ae_{12})^\alpha = 1+ae_{12}+a_{b}e_{2n}\). There is a similar argument involving \(1+a_{e_{n-1,n}}\). It is now clear that \(\alpha \in \mathcal{P}\).

It remains to be shown that if \(\alpha\) is any automorphism of \(G\) then we may (simultaneously) copy the effect of \(\alpha\) on \(H_2\) and on \(G/H_2\) using only the automorphisms of \([r], \mathcal{L}, \mathcal{D}\) and \(\mathcal{J}\).

Under an automorphism \(\alpha\) the subgroups \(A_i\) are either all left in-

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variant or are all reflected in the second diagonal. By multiplying by \( \tau \) if necessary we may assume that \( \alpha \) leaves each \( A_i \) invariant.

If \( (1+ae_{12})^\alpha = 1+\tilde{a}e_{12}+\cdots \) (where the extra terms are in \( N_{13} \)) then \( \{1+(ra+s\tilde{b})e_{12}\}^\alpha = 1+(\tilde{r}a+s\tilde{b})e_{12}+\cdots \) where \( r, s \) are integers mod \( p \). Hence \( \alpha \) induces a linear transformation \( a \rightarrow \tilde{a} \) of the vector space \( K \).

The set \( \{1+a_ie_{r,r+1}; (i=1, \cdots, k); (r=1, \cdots, n-1)\} \) is a minimal system of generators of \( G_n \). Hence \( \{(1+a_ie_{12})^\alpha; (i=1, \cdots, k)\} \) is part of a minimal system of generators of \( G_n \) and \( \tilde{a}_1, \cdots, \tilde{a}_k \) is again a basis of the vector space \( K \). The linear transformation \( a \rightarrow \tilde{a} \) is thus nonsingular.

If \( (1+ae_{23})^\alpha = 1+a'e_{23}+\cdots \), \( a \rightarrow a' \) is again a linear transformation of \( K \). Now the commutator \( (1+ae_{12}, 1+be_{23}) = 1+abe_{13} \) has the same value if we interchange \( a \) and \( b \), and so \( 1+ab'e_{13}+\cdots = 1+a'be_{13}+\cdots \). If \( b \neq 0 \), since \( N_{12} \) cannot map into \( N_{13} \) and \( N_{23} \) cannot map into \( N'_{23} \) neither \( b \) nor \( b' \) vanishes and \( \tilde{a}/b = a'/b' \). The effect of \( \alpha \) on \( P_{12} \) is thus the same as the effect on \( P_{23} \) apart from a constant factor. Since we may use a diagonal automorphism to give the required constant factors in \( P_{23}, P_{34}, \cdots, P_{n-1,n} \) there is an element \( \beta \) of \( J^D \) which has the same effect as \( \alpha \) on \( G_n \) mod \( H_2 \). Let us divide through by \( \beta \) and assume that \( \alpha \) induces the identity on \( G/H_2 \).

We now look for an inner automorphism which has the same effect as \( \alpha \) on \( H_2 \).

If, under \( \alpha \), \( 1+e_{23} \rightarrow 1+e_{23}+fe_{13}+ae_{24} \) (mod \( H_3 \)) then by commuting with \( 1+de_{12} \) we see that \( 1+de_{12} \rightarrow 1+de_{12}+dae_{14} \) (mod \( H_4 \)) (all \( d \in K \)). We transform by \( 1+ae_{34} \). This transformation also sends \( 1+e_{46} \rightarrow 1+e_{46}-ae_{26} \) (mod \( H_4 \)) and \( 1+e_{47} \rightarrow 1+e_{47}-ae_{37} \) (mod \( H_5 \)) but this is a necessary contribution since \( 1+e_{13}, 1+e_{46} \) commute and \( 1+e_{67} \rightarrow 1+e_{67} \) (mod \( H_2 \)).

If, under \( \alpha \), \( 1+e_{24} \rightarrow 1+e_{24}+be_{28}+ce_{14} \) (mod \( H_4 \)) we transform by \( 1+be_{46}-ce_{12} \). This transformation also affects \( 1+e_{67} \) and \( 1+e_{68} \) but here again there is a necessary contribution.

By such inner automorphisms using elements in \( P_{ij}, j-i=1 \), we copy the effect of \( \alpha \) on \( P_{ij} \) (\( j-i=2 \)) mod \( H_4 \) and \( P_{ij} \) (\( j-i=3 \)) mod \( H_6 \). Since \( H_2 \) is generated by the \( P_{ij} \) for which \( j-i=2, 3 \) we finally obtain an inner automorphism which has the same effect as \( \alpha \) on \( H_2 \) by transforming successively by elements in \( P_{ij}, j-i=1, 2, 3, \cdots \). This completes the proof of Theorem 8.