EXTREMAL PROBLEMS FOR STAR MAPPINGS

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The results obtained in §§1–2 are applied in §3 to show that the maximum possible value of $\Re\{B \log f'(z) + C \log [f(z)/z]\}$ at a point z_0 of the unit circle, for the class of functions $f(z) = z + \cdots$ which map $|z| < 1$ conformally onto star-shaped regions, is attained for a function $w = f(z)$ which maps the unit circle onto the w-plane with one or two radial slits. In §§4–5, we decide between these two alternatives in some cases.

1. A maximum problem. Let $\phi(z)$ and $\psi(z)$ be any two functions which are regular for $|z| \leqq 1$. We put

$$H = \sum_{\nu=1}^{n} \beta_\nu \phi(z_\nu) + \psi\left(\sum_{\nu=1}^{n} \beta_\nu z_\nu\right),$$

and consider the problem of maximizing $\Re H$. The points z_ν and the weights β_ν are to satisfy the conditions

$$|z_\nu| = 1, \qquad \beta_\nu > 0, \qquad \sum_{\nu=1}^{n} \beta_\nu = 1.$$

The existence of a maximum is clear if an upper bound for n is given. (First allow weights $\beta_\nu \geqq 0$, so that the variables range over a closed set. If some of the weights are zero at the maximum, the corresponding terms may be suppressed.) We shall show that $\Re H$ has a maximum even when no bound for n is given. Throughout, we shall let

$$\zeta = \sum_{\nu=1}^{n} \beta_\nu z_\nu.$$

THEOREM 1. *Suppose that the number of terms n and the weights β_ν are given. In order to maximize $\Re H$, all the points z_ν must satisfy the condition*

$$z[\phi'(z) + \psi'(\zeta)] \geqq 0.$$

PROOF. Suppose that the points z_ν have been chosen on the unit circle so as to maximize $\Re H$. This choice will also maximize $\Re H$ with respect to all choices of z_ν such that $|z_\nu| \leqq 1$. Holding all the other points fixed, and varying z_1, we see that

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$$dH=\beta_1[\phi'(z_1)+\psi'(\zeta)]dz_1.$$

No value of dz_1 directed into the unit circle can lead to dH with a positive real part. Thus if dz_1 is directed along an outer normal to the unit circle, the corresponding dH must be positive or zero. In other words, $[\phi'(z_1)+\psi'(\zeta)]z_1\geqq 0$.

COROLLARY 1. *In order that all the points z_ν coincide, it is sufficient that the transformation $w=z[\phi'(z)+\psi'(\zeta)]$ maps $|z|\leqq 1$ onto a star-shaped set in a one-to-one way.*

PROOF. The map is necessarily strictly star-shaped, and hence there is only one point on $|z|=1$ where $z[\phi'(z)+\psi'(\zeta)]\geqq 0$.

REMARK. These conditions are expressed in terms of an unknown quantity ζ. Thus to conclude from Corollary 1 that all the points z_ν actually coincide, so that $\Re H$ is maximized for $n=1$, we need to know that the map of $|z|\leqq 1$ by $w=z[\phi'(z)+\psi'(\zeta)]$ is star-shaped for every ζ in $|\zeta|\leqq 1$.

A function regular for $|z|\leqq 1$ maps this circle onto a star-shaped set in a one-to-one way if and only if $\Re[zf'(z)/f(z)]\geqq 0$ there and $f'(0)\neq 0$. Thus the condition required by Corollary 1 is that

$$\Re\left(1+\frac{z\phi''(z)}{\phi'(z)+\psi'(\zeta)}\right)\geqq 0$$

for $|z|\leqq 1$, and that $\phi'(0)+\psi'(\zeta)\neq 0$.

THEOREM 2. *Suppose that $\phi(z)$ is regular for $|z|\leqq 1$, but not of the form $az+b$. Then there is a number N, so that for any choice of $\psi(z)$, n, and the weights β_ν, the maximum value of $\Re H$ can be attained only when the number of distinct points z_ν does not exceed N.*

PROOF. Each of the points z_ν must lie on $|z|=1$, and satisfy the condition $z[\phi'(z)+\psi'(\zeta)]\geqq 0$. This condition has the form $z[\phi'(z)+K]\geqq 0$, where $K=\psi'(\zeta)$ is an unknown complex constant. In particular, $z[\phi'(z)+K]$ must be real at each of the points z_ν; we shall deduce the desired conclusion from this fact.

Using the condition stated in the above remark, we see that the function $z[\phi'(z)+K]$ will map $|z|\leqq 1$ onto a star-shaped set in a one-to-one way if $K=\psi'(\zeta)$ is sufficiently large in absolute value. In this case, $z[\phi'(z)+K]$ is positive at just one point and negative at just one point on $|z|=1$ (and thus, as noted in Corollary 1, all the points z_ν coincide.) It remains to consider what happens when $|K|$ is not so large.

Can $z[\phi'(z)+K]$ be identically real for $|z|=1$? If so, then applying the principle of maximum and minimum to the imaginary part of this function in $|z| \leqq 1$, we see that $z[\phi'(z)+K]$ is a constant. Putting $z=0$ shows that this constant is zero, and hence $\phi'(z) = -K$, so that $\phi(z) = -Kz+b$, contrary to hypothesis.

At how many points on $|z| = 1$ can $z[\phi'(z)+K]$ be real? This condition is equivalent to

$$z[\phi'(z) + K] = \bar{z}[\bar{\phi}'(\bar{z}) + \bar{K}]$$

or to

$$z[\phi'(z) + K] = z^{-1}[\bar{\phi}'(z^{-1}) + \bar{K}].$$

Since both sides are regular for $|z| = 1$, and they are not identically equal, there can be only a finite number of solutions.

How does the number of solutions of the above equation vary with K? Notice that a slight change in K can decrease, but cannot increase, the number of roots. It follows that the number of roots is bounded if K is bounded. Since there are exactly two roots when $|K|$ is large, the number of roots is therefore bounded for all values of K. Hence the number of distinct points z_ν is bounded, the bound depending only on the function $\phi(z)$.

Remark. If $\phi(z) = az+b$, then $H = \phi(\zeta) + \psi(\zeta)$. If H is not constant, then $\Re H$ is maximized only for $|\zeta| = 1$, and hence only when all the points z_ν coincide. An exceptional case occurs only when $\phi(z) = az+b$ and $\psi(z) = -az+c$. In this case, $H = b+c$, and the choice of the points z_ν is arbitrary.

Corollary 2. *The problem of maximizing $\Re H$ has a solution even when n is unrestricted.*

We now derive some conditions on the points z_ν on the assumption that we have maximized $\Re H$ without restricting n.

Theorem 3. *For the unrestricted problem, the maximum of*

$$\Re[\phi(z) + z\psi'(\zeta)]$$

on $|z| = 1$ is attained at each of the points z_ν.

Proof. It will be sufficient to prove this for $\nu = 1$. Let z_0 be any point on $|z| = 1$. For $0 \leqq \epsilon \leqq \beta_1$, we put

$$H(\epsilon) = \epsilon\phi(z_0) + (\beta_1 - \epsilon)\phi(z_1) + \sum_{\nu=2}^{n} \beta_\nu\phi(z_\nu)$$
$$+\psi\left(\epsilon z_0 + (\beta_1 - \epsilon)z_1 + \sum_{\nu=2}^{n} \beta_\nu z_\nu\right).$$

We see that $H(0)=H$, and

$$H'(0) = \phi(z_0) - \phi(z_1) + \psi'(\zeta)(z_0 - z_1).$$

By hypothesis, $\Re H(\epsilon) \leqq \Re H$, since $H(\epsilon)$ is a sum similar to H, but with $n+1$ terms. Hence $\Re H'(0) \leqq 0$, and therefore

$$\Re[\phi(z_0) + z_0\psi'(\zeta)] \leqq \Re[\phi(z_1) + z_1\psi'(\zeta)],$$

so that $\Re[\phi(z)+z\psi'(\zeta)]$ is maximized at z_1.

Remark. As the proof of Theorem 8 shows, the condition of Theorem 3 need not hold if we maximize the value of $\Re H$ with a preassigned value of n.

Corollary 3. *In order that all the points z_ν coincide, it is sufficient that the transformation $w=\phi(z)+z\psi'(\zeta)$ maps $|z| \leqq 1$ onto a convex set in a one-to-one way.*

Remark. It is known that an analytic function $F(z)$ defines a one-to-one convex mapping of the unit circle if and only if $zF'(z)$ defines a one-to-one star mapping. Hence the conditions of Corollaries 1 and 3 are actually identical.

Theorem 4. *For the unrestricted problem, the expression*

$$z[\phi'(z) + \psi'(\zeta)]$$

is real, not only at the points z_ν, but also at additional points on $|z|=1$ separating each pair of such points.

Proof. Consider the curve

$$w = \phi(z) + z\psi'(\zeta), \qquad |z| = 1.$$

The outer normal at any point has the direction

$$\text{amp}\,\{z[\phi'(z) + \psi'(\zeta)]\},$$

provided $\phi'(z)+\psi'(\zeta)\neq 0$. Thus at any relative maximum or minimum of $\Re[\phi(z)+z\psi'(\zeta)]$ on $|z|=1$, the quantity $z[\phi'(z)+\psi'(\zeta)]$ must certainly be real. Since the relative maxima and minima must alternate, and the points z_ν are among the relative maxima by Theorem 3, we have the required result. Notice also that at the points z_ν, which are in fact absolute maxima, we must have $z[\phi'(z)+\psi'(\zeta)]\geqq 0$, as stated in Theorem 1.

Remark. It is not clear whether or not the condition of Theorem 4 also holds when the value of n is preassigned.

2. **An example.** The following example serves as an illustration of the preceding theorems, and also provides results which will be used

in later sections. Let

$$\phi(z) = A \log (1 + rz), \qquad \psi(z) = B \log (1 + sz),$$

where $0<r<1$, $0<s<1$, and A and B are complex numbers, not both zero. In this case,

$$H = A \sum_{\nu=1}^{n} \beta_\nu \log (1 + rz_\nu) + B \log\left(1 + s \sum_{\nu=1}^{n} \beta_\nu z_\nu\right),$$

and we are to maximize $\Re H$ subject to the usual conditions. Now according to Theorem 4, the number of distinct points z_ν is at most half the number of points on $|z| = 1$ where

$$h(z) = z\left(\frac{Ar}{1 + rz} + \frac{Bs}{1 + s\zeta}\right)$$

is real. We must estimate the number of such points, without knowing the value of ζ. As in the proof of Theorem 2, the condition that $h(z)$ is real on the unit circle reduces to

$$z\left(\frac{Ar}{1 + rz} + K\right) = \frac{\bar{A}r}{z + r} + \frac{\bar{K}}{z},$$

where $K = \psi'(\zeta) = Bs/(1+s\zeta)$. Clearing of fractions, we have

$$z^2(z + r)[Ar + K(1 + rz)] = (1 + rz)[\bar{A}rz + \bar{K}(z + r)].$$

This is an algebraic equation in z of at most the fourth degree, but not an identity. Hence it has at most four roots on $|z| = 1$. It follows that there are at most two distinct values of z_ν. Thus the maximum possible value of $\Re H$ is attained for $n \leqq 2$, and essentially only in this case.

There are many cases in which we can prove that all the points z_ν coincide. In the first place, this is clearly true if $A = 0$. Using the fact that the function $\log (1+rz)$ maps $|z| \leqq 1$ onto a convex set, we see that the same is true if $B = 0$. Suppose now that $A \neq 0$ and $B \neq 0$. We shall derive a simple sufficient condition which depends, besides on the values of r and s, only on the amplitude of A/B.

According to Corollary 1, it is sufficient to show that for any ζ with $|\zeta| \leqq 1$, the function $h(z)$ considered above maps $|z| \leqq 1$ onto a star-shaped set in a one-to-one way. This requires that $\Re[zh'(z)/h(z)] \geqq 0$ for $|z| \leqq 1$, and that $h'(0) \neq 0$. Now

$$\frac{zh'(z)}{h(z)} = \frac{Ar + K(1 + rz)^2}{Ar(1 + rz) + K(1 + rz)^2}.$$

The reciprocal of this fraction may be written in the form

$$1 + \frac{rz}{1+q}, \quad \text{where} \quad q = \frac{K(1+rz)^2}{Ar} = \frac{Bs(1+rz)^2}{Ar(1+s\zeta)}.$$

This quantity will certainly lie in the right half-plane if $|1+q| \geqq r$ which in turn will hold if $|\text{amp } q| \leqq \pi - \text{arc sin } r$, where the principal value of the amplitude is taken. Now

$$\text{amp } q = 2 \text{ amp } (1 + rz) - \text{amp } (1 + s\zeta) - \text{amp } (A/B).$$

Since $|\text{amp } (1+rz)| \leqq \text{arc sin } r$ and $|\text{amp } (1+s\zeta)| \leqq \text{arc sin } s$, it will be sufficient to have

$$|\text{amp } (A/B)| + 3 \text{ arc sin } r + \text{arc sin } s \leqq \pi.$$

Since this implies $|1+q| \geqq r$, it also yields $h'(z) = Ar(1+q)/(1+rz)^2 \neq 0$. It is therefore a sufficient condition for the coincidence of all the points z_ν. Unless $A/B<0$, it will hold at least when r and s are sufficiently small.

3. Star mappings: the main theorem. We consider functions $f(z) = z + \cdots$ which are regular and univalent for $|z| < 1$, and which map the unit circle onto star-shaped regions. We start by showing how such a function can be approximated by a function which maps the unit circle onto the w-plane with a finite number of radial slits. The possibility of such an approximation follows from general theorems on mapping variable regions, but we prefer a method which also yields a formula for the approximating functions.

It will be sufficient to consider an admissible function $f(z)$ which is regular for $|z| \leqq 1$. Let $g(z) = zf'(z)/f(z) = 1 + \cdots$, which is also regular for $|z| \leqq 1$. We have $\Re g(z) \geqq 0$ there. By a form of the Poisson integral formula (compare [**4**, p. 2]), we have

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} \Re g(e^{i\theta}) \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} d\theta.$$

Hence

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{1}{2\pi} \int_0^{2\pi} \Re g(e^{i\theta}) \frac{2e^{-i\theta}}{1 - ze^{-i\theta}} d\theta,$$

and therefore

$$\log \frac{f(z)}{z} = \frac{1}{2\pi} \int_0^{2\pi} \Re g(e^{i\theta}) 2 \log \frac{1}{1 - ze^{-i\theta}} d\theta.$$

Approximating the integral by a sum shows that the given function can be approximated by a function for which

$$\log \frac{f(z)}{z} = 2 \sum_{\nu=1}^{n} \beta_\nu \log \frac{1}{1 - ze^{-i\theta_\nu}},$$

that is, by a function of the form

$$f(z) = \frac{z}{\prod_{\nu=1}^{n} (1 - ze^{-i\theta_\nu})^{2\beta_\nu}},$$

where $\beta_\nu > 0$ and $\beta_1 + \cdots + \beta_n = 1$. An alternative derivation is given by Goodman [1, p. 280].

We may suppose that the numbers θ_ν are distinct. It is easily seen that amp $f(z)$ is constant on each of the arcs into which the points $e^{i\theta_\nu}$ divide the unit circle, and has a discontinuity of $2\pi\beta_\nu$ at $e^{i\theta_\nu}$. Thus $f(z)$ maps the unit circle onto the w-plane with n radial slits, the angles between the slits being $2\pi\beta_\nu$. The length of the slits is governed by the distribution of the points $e^{i\theta_\nu}$; when consecutive points are close together, the intermediate arc must go into a slit whose end is far from the origin.

THEOREM 5. *Among the normalized functions $f(z)$ which map the unit circle conformally onto star-shaped regions, the maximum of*

$$\Re \left\{ B \log f'(z) + C \log \frac{f(z)}{z} \right\},$$

where B and C are complex numbers (not both zero), at any point z_0 with $0 < |z_0| < 1$, is attained for a function which maps the unit circle onto the w-plane with at most two radial slits. Every extremal function must be of this type.

REMARK. Notice that $f'(z) = 1 + \cdots$ and $f(z)/z = 1 + \cdots$ are both different from zero throughout the unit circle, so that $\log f'(z)$ and $\log [f(z)/z]$ have no singularities there; in each case, a single-valued function which is regular in the unit circle is determined by choosing the branch which vanishes at the origin.

PROOF. Consider first the above functions which map the unit circle onto the w-plane with a finite number of radial slits. We shall reduce the present extremal problem to the maximum problem studied in §§1–2. If $|z_0| = r$, then the transformation $\tau = 1/(1 - z_0 t)$ maps $|t| = 1$ onto the circle having $1/(1+r)$ and $1/(1-r)$ as ends of a diameter; that is, onto the circle

$$\left|\tau - \frac{1}{1-r^2}\right| = \frac{r}{1-r^2}.$$

Thus if we put

$$\frac{1}{1-z_0e^{-i\theta_\nu}} = \frac{1+rz_\nu}{1-r^2},$$

we shall have $|z_\nu| = 1$. Making this substitution, we find that

$$\frac{f(z_0)}{z_0} = \prod_{\nu=1}^{n}\left(\frac{1}{1-z_0e^{-i\theta_\nu}}\right)^{2\beta_\nu} = \frac{1}{(1-r^2)^2}\prod_{\nu=1}^{n}(1+rz_\nu)^{2\beta_\nu},$$

and

$$\begin{aligned}\frac{z_0f'(z_0)}{f(z_0)} &= 1 + 2\sum_{\nu=1}^{n}\frac{\beta_\nu z_0e^{-i\theta_\nu}}{1-z_0e^{-i\theta_\nu}} = 2\sum_{\nu=1}^{n}\frac{\beta_\nu}{1-z_0e^{-i\theta_\nu}} - 1\\ &= \frac{1+r^2}{1-r^2} + \frac{2r}{1-r^2}\sum_{\nu=1}^{n}\beta_\nu z_\nu = \frac{1+r^2}{1-r^2}\left(1 + s\sum_{\nu=1}^{n}\beta_\nu z_\nu\right),\end{aligned}$$

where $s = 2r/(1+r^2)$. Hence

$$\begin{aligned}B\log f'(z_0) + C\log\frac{f(z_0)}{z_0} &= B\log\frac{z_0f'(z_0)}{f(z_0)} + (B+C)\log\frac{f(z_0)}{z_0}\\ &= H + A\log\frac{1}{1-r^2} + B\log\frac{1+r^2}{1-r^2},\end{aligned}$$

where $A = 2(B+C)$ and

$$H = A\sum_{\nu=1}^{n}\beta_\nu\log(1+rz_\nu) + B\log\left(1 + s\sum_{\nu=1}^{n}\beta_\nu z_\nu\right).$$

Hence the problem under consideration reduces to maximizing $\Re H$. Since the points z_ν are arbitrary points on $|z| = 1$, the desired result follows from §2.

It remains to consider star mappings in general. The previous extremal functions still furnish a maximum. Also, there cannot be any additional extremal functions. For the quantity H above cannot be near its maximum unless all the weights β_ν are small except when z_ν is near one or the other of two points on $|z| = 1$. Thus any extremal function can be approximated by a function which maps $|z| < 1$ onto the w-plane with a finite number of slits, all of which (in view of the remarks preceding Theorem 5) are either near to one or the other of two rays, or else start far from the origin. Hence the extremal function must be of the stated type.

REMARK. Stroganoff [5] proved the above theorem in the special case $B=\pm i$, $C=0$, and indeed by deriving (though by a different method) conditions equivalent to ours, suitably specialized. He was able to show that, in the case considered, only one slit was actually required; thus the maximum and minimum values of amp $f'(z)$ for normalized star mappings are attained by functions of the form $f(z)=z/(1-ze^{-i\theta})^2$. We shall not prove this result, but we do give in §4 certain other cases where we must have just one slit, and in §5 an instance where two slits are required.

4. **Some cases with one slit.** According to Theorem 5, the maximum value of $\Re\{B \log f'(z)+C \log [f(z)/z]\}$ for normalized star mappings is attained by a function which maps the unit circle onto the w-plane with at most two radial slits. If $B=0$, using a result noticed in §2, it is seen that only one slit is required; compare Marx [3, Satz B]. If $B\neq 0$, we may, without loss of generality, suppose that $B=e^{-i\alpha}$ and $C=\lambda e^{-i\alpha}$. A partial solution to the problem whether one slit is sufficient in this case is given by the following theorem.

THEOREM 6. *Among the normalized functions $f(z)$ which map the unit circle onto star-shaped regions, the maximum of*

$$\Re\{e^{-i\alpha}[\log f'(z) + \lambda \log (f(z)/z)]\}$$

at a point z_0 with $|z_0|=r$, where $0<r<1$, can be attained only for a function which maps the unit circle onto the w-plane with one radial slit, provided that

$$|\text{amp } (1+\lambda)| + 3 \text{ arc sin } r + 2 \text{ arc tan } r \leqq \pi.$$

PROOF. By the same procedure as in the proof of Theorem 5, we can reduce this to a result proved in §2. With the notation previously used, $A/B=2(1+\lambda)$. Since $s=2r/(1+r^2)$, we see that

$$\text{arc sin } s = 2 \text{ arc tan } r.$$

It remains only to apply the result at the end of §2.

REMARK. Thus the conclusion of the theorem holds at least for small values of r, unless $\lambda\leqq -1$. For $\lambda=-1$, it is easily seen that the conclusion actually holds for all values of r. If $\lambda>-1$, the condition of the theorem reduces to $3 \text{ arc sin } r+2 \text{ arc tan } r\leqq\pi$; hence the conclusion holds in this case at least for $r\leqq 0.62$. On the other hand, for $\lambda=-2$, the result does not hold even for small values of r; compare §5.

THEOREM 7. *For the class of normalized functions $f(z)$ which map the unit circle onto star-shaped regions, the set of possible values of*

$\log f'(z)$ *at a point* z_0 *with* $|z_0| = r$ *is exactly the map of* $|z| \leqq r$ *by the function* $h(z) = \log [(1+z)/(1-z)^3]$, *at least if* $r \leqq 0.6$.

PROOF. We show first that $h(z)$ maps $|z| \leqq r$ onto a convex set. Since

$$h'(z) = \frac{1}{1+z} + \frac{3}{1-z},$$

we see that $\Re h'(z) > 0$, so that $h(z)$ is univalent. We find that

$$1 + \frac{zh''(z)}{h'(z)} = \frac{2(1+z+z^2)}{(1-z^2)(2+z)} = \frac{2(1-z^3)}{(1-z)(1-z^2)(2+z)}\cdot$$

Hence

$$\left|\operatorname{amp}\left(1 + \frac{zh''(z)}{h'(z)}\right)\right| \leqq \arcsin r + \arcsin r^2 + \arcsin r^3 + \arcsin \frac{r}{2}.$$

The right side is less than $\pi/2$ for $r \leqq 0.6$, which shows that the map of $|z| \leqq r$ by $h(z)$ is convex.

It is easily seen that the set of possible values of $\log f'(z)$ at a point z_0 on $|z| = r$ includes the map of $|z| \leqq r$ by $h(z)$; we need consider only functions of the form $f(z) = z/(1-kz)^2$ with $|k| \leqq 1$. We have just shown that this map is a convex set. Now by Theorem 6, with $\lambda = 0$, the maximum of $\Re[e^{-i\alpha} \log f'(z_0)]$ is attained for $f(z) = z/(1-ze^{-i\theta})^2$ for some real θ, and hence is equal to the maximum of $\Re[e^{-i\alpha}h(z)]$ on $|z| = r$. Hence the possible values of $\log f'(z_0)$ lie in the map of $|z| \leqq r$ by $h(z)$.

REMARK. Using the notation of [4], the hypothesis that $f(z)$ is a star mapping may be written

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \qquad \text{for } |z| < 1,$$

and the conclusion of Theorem 7 is

$$\log f'(z) \prec \log \frac{1+z}{(1-z)^3} \qquad \text{for } |z| \leqq 0.6.$$

This conclusion may also be written

$$f'(z) \prec \frac{1+z}{(1-z)^3} \qquad \text{for } |z| \leqq 0.6.$$

It was conjectured by Marx [3, p. 66], that Theorem 7 holds in the entire unit circle. This was proved by Marx for $r \leqq 2-3^{1/2} = 0.267 \cdots$; in [4, §3.8], the bound was increased to $(5-17^{1/2})/2 = 0.438 \cdots$. The present result is a further improvement.

If the conjecture of Marx is correct, then Theorem 6, with $\lambda = 0$, will also hold for all r. It should be noted that for $\alpha = 0$ or π, this follows from classical results on univalent functions. Also, for $\alpha = \pm\pi/2$, it was proved by Stroganoff [5] and Goodman [1], but their methods do not seem to be applicable to other values of α.

The bounds given for r in Theorems 6 and 7 could be improved by making more careful estimates. However, we cannot prove Theorem 6 with $\lambda = 0$ for the whole unit circle on the basis of Corollary 3, since (as is easily seen) the required mappings are not all convex. Equally, this result cannot follow from Corollary 1. Also, admitting Theorem 6 with $\lambda = 0$ for all r, we still could not deduce Theorem 7 for the whole unit circle, since the function $h(z)$ used there is also not convex for $|z| < 1$. Thus a new method is required to make a complete proof of the conjecture of Marx.

5. **Some cases with two slits.** It is known that, for the class of functions $f(z) = z + \cdots$ which are regular and univalent for $|z| < 1$, we have

$$1 - r^2 \leqq \left| \frac{z^2 f'(z)}{[f(z)]^2} \right| \leqq \frac{1}{1 - r^2} \qquad \text{for } |z| = r.$$

(This result is equivalent to finding bounds for the derivative of a function univalent in the exterior of the unit circle and normalized at ∞; see Löwner [2, Satz IV and Satz V].) The lower bound is attained at $z = r$ for a function which maps $|z| < 1$ onto the w-plane with slits at either or both ends of the real axis. The extremal functions for the narrower class of star mappings are the same. This shows that Theorem 6 does not hold for $\lambda = -2$ and $\alpha = \pi$, even for small values of r: the extremal functions do not necessarily map onto the plane with just one slit.

The upper bound in the above inequality is also attained for a function which maps the unit circle onto the w-plane slit along one or both ends of a straight line; but in this case, the line does not pass through the origin, so that the extremal functions do not furnish star mappings. The upper bound for the class of star mappings is consequently smaller. Although we shall not calculate this maximum, we shall show that it can be attained only for a function which maps the unit circle onto the w-plane with exactly two radial slits. Hence in Theorem 6, for $\lambda = -2$ and $\alpha = 0$, we must have two slits, even for small values of r. Thus there are cases where we must have one slit, cases where we must have two slits, and cases where both alternatives are possible.

THEOREM 8. *The maximum value of* $|z^2f'(z)/[f(z)]^2|$ *at a point* $z_0\neq 0$ *in the unit circle, for the class of normalized star mappings, can be attained only for a function which maps the unit circle onto the* w-*plane with exactly two radial slits.*

PROOF. It will be sufficient, in view of Theorem 5, to show that the maximum cannot be attained if the map has just one radial slit. When the map has just one radial slit, we have

$$f(z) = \frac{z}{(1 - ze^{-i\theta})^2}, \quad \text{hence} \quad \frac{z^2f'(z)}{[f(z)]^2} = 1 - z^2e^{-2i\theta}.$$

It follows that, in this case,

$$1 - r^2 \leqq \left|\frac{z^2f'(z)}{[f(z)]^2}\right| \leqq 1 + r^2 \qquad \text{for } |z| = r.$$

At $z=r$, the minimum is furnished for $\theta=0$ or π, in agreement with results mentioned above, and the maximum is furnished for $\theta=\pm\pi/2$. We are to show that the latter case does not furnish the maximum for the whole class of star mappings. For this purpose, we shall transform the problem to the form considered in §1, and show that the condition of Theorem 3 is not satisfied.

The given problem is to maximize

$$\Re\{\log f'(z) - 2\log [f(z)/z]\}.$$

We may, without loss of generality, restrict our attention to the point $z=r$, where $0<r<1$. As in the proof of Theorem 5, this problem is reduced to maximizing $\Re H$, where

$$H = -2\sum_{\nu=1}^{n}\beta_\nu \log (1 + rz_\nu) + \log\left(1 + s\sum_{\nu=1}^{n}\beta_\nu z_\nu\right),$$

where $s=2r/(1+r^2)$, the points z_ν are to satisfy $|z_\nu|=1$, and the weights β_ν are subject to the usual conditions.

We are to show that $\Re H$ is not maximized for $n=1$. Recalling how the points z_ν were introduced in the proof of Theorem 5, we see that the maximum possible value of $\Re H$ for $n=1$ is attained for z_1 satisfying

$$\frac{1}{1 - ri} = \frac{1 + rz_1}{1 - r^2},$$

that is, for

$$z_1 = \frac{-2r + (1 - r^2)i}{1 + r^2}.$$

We can check that the condition of Theorem 1 is satisfied, if we also substitute the same value for ζ; we must put $\phi(z) = -2 \log(1+rz)$ and $\psi(z) = \log(1+sz)$. On the other hand, we shall show that the condition of Theorem 3 is not satisfied.

If the maximum value of $\Re H$ is attained when $n=1$, we may take

$$\zeta = \frac{-2r + (1 - r^2)i}{1 + r^2}.$$

According to Theorem 3, the maximum of

$$P(z) = \Re\left(-2 \log(1 + rz) + \frac{sz}{1 + s\zeta}\right)$$

on $|z| = 1$ should be attained at $z=\zeta$. Now for $|z| = 1$, we have

$$|1 + rz|^2 = (1 + rz)(1 + r\bar{z}) = 1 + r^2 + 2rx,$$

where $z = x + iy$. Also, a simple calculation shows that

$$\frac{s}{1 + s\zeta} = \frac{2r}{1 + r^2} - \frac{4r^2 i}{1 - r^4}.$$

It is thus clearly sufficient to consider the case $y \geqq 0$, so that $z = x + i(1-x^2)^{1/2}$. Putting $P(z) = p(x)$, we find that

$$p(x) = \frac{2rx}{1 + r^2} + \frac{4r^2(1 - x^2)^{1/2}}{1 - r^4} - \log(1 + r^2 + 2rx),$$

hence

$$p'(x) = \frac{4r^2 x}{1 + r^2}\left(\frac{1}{1 + r^2 + 2rx} - \frac{1}{(1 - r^2)(1 - x^2)^{1/2}}\right).$$

Now the second factor on the right is never positive; indeed

$$(1 + r^2 + 2rx)^2 - [(1 - r^2)(1 - x^2)^{1/2}]^2 = [2r + (1 + r^2)x]^2.$$

This shows that $p'(x) > 0$ for $x < 0$, and $p'(x) < 0$ for $x > 0$, except that $p'(x) = 0$ not only for $x = 0$, but also for $x = -2r/(1+r^2)$. The latter point, which corresponds to $z = \zeta$, does not, however, furnish a maximum. On the contrary, the maximum of $P(z)$ for $|z| \leqq 1$ occurs only at $z = i$. Thus the condition of Theorem 3 is not satisfied, so that $\Re H$ is not maximized for $n = 1$.

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CRITERIA OF BOUNDEDNESS OF THE SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

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1. In this paper we shall use the following lemma to derive some criteria of boundedness of the solutions of certain nonlinear differential equations.

Lemma. *Suppose that the following conditions are satisfied*:

1. $u(x)$ *and* $v(x)$ *are real-valued functions, defined and non-negative for* $x \geqq a$,

2. $v(x)$ *and* $u(x)v(x)$ *belong to* $L(a, R)$ *for every* $R > a$,

3. *for some positive constant* M

$$u(x) \leqq M + \int_a^x u(t)v(t)dt \qquad (x \geqq a). \tag{1.1}$$

Then

$$u(x) \leqq M \exp\left(\int_a^x v(t)dt\right) \qquad (x \geqq a). \tag{1.2}$$

This lemma is useful in the study of boundedness and asymptotic behavior of the solutions of differential equations, see for instance [**1**; **2**; **3**]. Its proof is simple: divide the left-hand member of (1.1) by its right-hand member and multiply the result by $v(x)$, then integrate

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