THE BERNSTEIN APPROXIMATION PROBLEM[1]

HARRY POLLARD

Let $k(x)$ be a complex-valued function continuous on the interval $(-\infty, \infty)$, and satisfying the condition

$$\text{(i)} \qquad \lim x^n k(x) = 0, \qquad x \to \pm\infty, n = 0, 1, \cdots .$$

The Bernstein problem calls for necessary and sufficient conditions in order that the set of functions $x^n k(x)$, $n=0, 1, \cdots$, form a fundamental set[2] in the linear space C_0 of functions continuous on $(-\infty, \infty)$, vanishing at $\pm\infty$, and normed by the maximum.

A few simple observations help to reduce this formulation of the problem to a more accessible form. First, if $x^n k(x)$ is fundamental, then $k(x)$ can have no zeros. For each element in the closure of the finite linear combinations of $\{x^n k(x)\}$ must inherit any zero of $k(x)$. Secondly $x^n k(x)$, $n=0, 1, \cdots$, can be fundamental if and only if $x^n|k(x)|$ is also fundamental. This is true because a function $f(x)$ in C_0 is approximable by linear combinations of the $x^n k(x)$ if and only if the continuous function[3] $f(x)$ sgn $k(x)$ is approximable by the same linear combinations of $x^n|k(x)|$. Consequently it may be assumed that $k(x)=1/\Phi(x)$, where

$$\text{(ii)} \qquad \lim x^n/\Phi(x) = 0, \quad x \to \pm\infty, n = 0, 1, \cdots ;$$

$$\text{(iii)} \qquad \Phi(x) > 0, \qquad -\infty < x < \infty.$$

Finally, in view of (iii) we may restrict ourselves to *real* space C_0.

For functions $\Phi(x)$ which in addition to (ii) and (iii) have the properties that $\Phi(x)=\Phi(-x)$ and that $\Phi(x)$ is increasing for $x>0$, necessary and sufficient conditions have been found by Bernstein [3] and by Ahiezer and Bernstein [2]. An independent solution of the same nature as these was found by the author [5], but without the additional restrictions on Φ.

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[2] Following Banach, *Théorie des opérations linéaires*, p. 58, I call a set S in a Banach space B *fundamental* if the set of finite linear combinations of the elements of S is dense in B. By now the words "closed" and "complete" as well as their French or German equivalents have become thoroughly confused.

[3] Continuous because $k(x) \neq 0$.

The purpose of this paper is to present a criterion simpler than any of these, and which requires only the original restrictions (ii) and (iii) on Φ. In §4 it will be shown how this result can be converted into a solution of the corresponding problem for the interval $(0, \infty)$. In §5 the solution will be extended to the spaces $L^p(-\infty, \infty)$, $p \geqq 1$.

The main result to be proved is this:

THEOREM 1. *In order that the set of functions* $x^n/\Phi(x)$, $n=0, 1, \cdots$, *subject to conditions* (ii) *and* (iii) *be fundamental in* C_0 *it is necessary and sufficient that*[4]

$$\text{(1)} \qquad \text{u. b.} \int \frac{\log^+ |p(x)|}{1+x^2}\, dx = \infty,$$

where the upper bound is taken over all real polynomials p *for which* $|p(x)| < \Phi(x)$, $-\infty < x < \infty$.

1. **The necessity.** To prove the necessity we begin by borrowing a lemma of Ahiezer and Babenko [1]. It is reproduced here because their paper is not readily available, and in a revised form useful in treating the L^p problem (§5).

LEMMA 1.1. *Let* $a(x)$ *be a function measurable on* $(-\infty, \infty)$, *such that* $a(x) > 0$ *a. e. and such that* $x^n a(x) \in L(-\infty, \infty)$, $n=0, 1, \cdots$. *If*

$$\text{(1.1)} \qquad \int \frac{|\log a(x)|}{1+x^2}\, dx < \infty,$$

then there exists a real function $b(x) \not\equiv 0$, *bounded and belonging to* L^p *for every* $p \geqq 1$ *such that*

$$\text{(1.2)} \qquad \int x^n a(x) b(x)\, dx = 0, \qquad n = 0, 1, \cdots.$$

Since $a(x)$ belongs to L and (1.1) is assumed to hold, a theorem of Hille and Tamarkin [4] guarantees the existence of a (complex-valued) function $G(x)$ such that $|G(x)| = a(x)$ and

$$\int e^{-ixu} G(x) dx = 0, \qquad u < 0.$$

Since also

$$\int e^{-ixu} \frac{dx}{(1-ix)^2} = 0, \qquad u < 0,$$

the convolution theorem yields

[4] Limits will be omitted from doubly infinite integrals. $\log^+ x$ is defined here as $\max(0, \log x)$.

$$\int e^{-ixu} \frac{G(x)}{(1-ix)^2} dx = 0, \qquad u < 0. \tag{1.3}$$

Now $x^n G(x) = x^n |a(x)|$ is in L for each n, by hypothesis. Consequently we may differentiate under the integral (1.3) and obtain from the continuity at $u=0$ that

$$\int x^n \frac{G(x)}{(1-ix)^2} dx = 0, \qquad n = 0, 1, \cdots.$$

This may be written as

$$\int x^n a(x) \frac{\operatorname{sgn} G(x)}{(1-ix)^2} dx = 0, \qquad n = 0, 1, \cdots$$

(1.2) is now established, with

$$b(x) = \text{real part of } \frac{\operatorname{sgn} G(x)}{(1-ix)^2}.$$

To prove the necessity of condition (1) suppose that $x^n/\Phi(x)$, $n=0, 1, \cdots$, is fundamental in C_0, and that

$$\text{u.b.} \int \frac{\log^+ |p(x)|}{1+x^2} dx < \infty \tag{1.4}$$

for all polynomials p such that $|p(x)| < \Phi(x)$. We shall obtain a contradiction from these assumptions.

Let $c_n = 1 - 1/n$. For each $n > 1$ the function $c_n/(1+x^2)$ belongs to C_0. Consequently for each n there exists a finite linear combination $p_n(x)/\Phi(x)$ of the functions $x^n/\Phi(x)$ such that

$$\left| \frac{p_n(x)}{\Phi(x)} - \frac{c_n}{1+x^2} \right| < \frac{1}{n}.$$

Clearly $|p_n(x)| < \Phi(x)$. Moreover

$$\lim_{n\to\infty} p_n(x) = \frac{\Phi(x)}{1+x^2}.$$

From this, (1.4), and Fatou's lemma it follows that

$$\int \frac{\log^+ |\Phi(x)|}{1+x^2} dx < \infty. \tag{1.5}$$

Because $x^2/\Phi(x) = O(1)$, $x \to \pm\infty$, the function $1/\Phi(x) \in L$. Therefore (1.5) is equivalent to

$$\int \frac{|\log \Phi(x)|}{1+x^2}\, dx < \infty.$$

By Lemma 1.1 with $a(x) = 1/\Phi(x)$ there exists a function $b(x) \not\equiv 0$ in L such that

$$\int \frac{x^n}{\Phi(x)} b(x) dx = 0.$$

In other words, there is a linear functional on C_0 which vanishes for each element $x^n/\Phi(x)$ but is not identically zero. This contradicts the hypothesis that $x^n/\Phi(x)$, $n=0, 1, \cdots$, is fundamental.

2. **Lemmas.** The following lemmas are probably familiar to workers in the field, but do not seem to be recorded.

LEMMA 2.1. *There exist absolute constants K, L, and y_0 such that if*

$$H(z) = \int \frac{d\sigma(u)}{z-u}, \qquad z = x+iy,\ y>0, \tag{2.1}$$

and σ is a real function of bounded variation, then[5]

$$\int \frac{|H(x+iy)|^{1/2}}{1+x^2}\, dx < K\{V(\sigma) + (2V(\sigma))^{1/2} + L\}, 0 < y \leqq y_0. \tag{2.2}$$

First suppose that σ is increasing. Then an argument of Titchmarsh [7, pp. 144–145] shows that (2.2) holds without the factor 2 in the second term on the right-hand side. In the general case write $\sigma = \sigma_1 - \sigma_2$ where σ_1 and σ_2 are increasing and $V(\sigma) = V(\sigma_1) + V(\sigma_2)$. Apply the preceding remark to each part separately and add. (2.2) now follows with the factor 2 in the second term.

LEMMA 2.2. *If $H(z)$ is defined as in the preceding lemma and $V(\sigma) \leqq 1$, then*

$$\int \frac{\log^+ |H(x+iy)|}{1+x^2}\, dx < A, \qquad 0 < y \leqq y_0, \tag{2.3}$$

where A and y_0 are absolute constants.

Since $\log^+ x \leqq x^{1/2}$ this follows from the preceding lemma with $A = K(1+2^{1/2}+L)$.

LEMMA 2.3. *If $H(z)$ is defined by (2.1) where σ is real and not substantially a constant, then for some constant C independent of y and some $y_1 > 0$*

[5] $V(\sigma)$ denotes the total variation of σ.

$$\int \frac{|\log |H(x+iy)||}{1+x^2}\,dx < C, \qquad 0 < y \leqq y_1. \tag{2.4}$$

Note that there is no claim that C and y_1 are *absolute* constants. It follows from the hypotheses that $H(z)$ does not vanish identically for $y>0$. For if it did it would also vanish for $y<0$ by conjugacy, and then σ would be substantially a constant, by Stieltjes' inversion of (2.1).

Since $|\log x| = 2 \log^+ x - \log x$, (2.4) follows from (2.3) and the following lemma.

LEMMA 2.4. *Let $H(z)$ be a function analytic for $y>0$, bounded in each half-plane $y \geqq \eta > 0$, and let it not vanish identically. Then for some constants M and y_2*

$$\int \frac{\log |H(x+iy)|}{1+x^2}\,dx \geqq M > -\infty, \qquad 0 < y \leqq y^2. \tag{2.5}$$

CASE (i). $H(i) \neq 0$. For each $\eta > 0$ define

$$h_\eta(w) = H(z + i\eta),$$

where $z = i(1-w)/(1+w)$, $w = re^{i\theta}$, maps the half-plane $y \geqq 0$ onto the disk $|w| \leqq 1$. Let the disks C_1 and C_2 be respectively the images of the regions $y \geqq 0$, $y \geqq -\eta/2$. Then $C_2 \supset C_1$, the boundaries of C_1 and C_2 being tangent at $w = -1$. Since $H(z+i\eta)$is bounded and analytic for $y \geqq -\eta/2$ it follows that $h_\eta(w)$ is analytic in C_1 except possibly at $w = -1$, is bounded on C_1, and vanishes on at most a countable set on the boundary of C_1, i.e. $|w| = 1$.

Now $h_\eta(0) = H(i(1+\eta))$. Since $H(i) \neq 0$ it follows that for sufficiently small choice of η, say $0 < \eta \leqq \eta_0$, we have $|h_\eta|(0) \geqq \delta > 0$. By Jensen's theorem [6, p. 125]

$$\int_0^{2\pi} \log |h_\eta(re^{i\theta})|\, d\theta > 2\pi \log |h_\eta(0)| > 2\pi \log \delta,$$

provided $0 \leqq r < 1$, $0 < \eta \leqq \eta_0$. Since h_η is bounded in C_1 for each η and since

$$\lim_{r \to 1} \log |h_\eta(re^{i\theta})| = \log |h_\eta(e^{i\theta})|$$

holds except on at most a countable set, it follows from Fatou's lemma that

$$\int_0^{2\pi} \log |h_\eta(e^{i\theta})| d\theta \geqq 2\pi \log \delta, \qquad 0 < \eta \leqq \eta_0.$$

Since

$$d\theta = 2(1 + x^2)^{-1}dx,$$

(2.5) follows from this on mapping back into the z-plane, with $y_2 = \eta_0$ and $M = 2\pi \log \delta$.

CASE (ii). If $H(i) = 0$ then for some choice of the integer p the function $H_1(z) = (z-i)^{-p}H(z)$ does not vanish at i and fulfills the conditions of Case (i). (2.5) then holds for $H_1(z)$, and then for $H(z)$ on readjustment of the constants.

3. **The sufficiency of condition** (1). If $x^n/\Phi(x)$, $n = 0, 1, \cdots$, is *not* fundamental then there exists a real function of bounded variation on $(-\infty, \infty)$, not substantially a constant, such that

$$\int \frac{u^n}{\Phi(u)} d\sigma(u) = 0, \qquad n = 0, 1, \cdots. \tag{3.1}$$

It may be assumed that $V(\sigma) \leqq 1$. Define

$$H(z) = \int \frac{1}{\Phi(u)} \frac{d\sigma(u)}{z - u}. \tag{3.2}$$

From the identity

$$\frac{1}{z-u} = \frac{1}{z} + \frac{u}{z^2} + \cdots + \frac{u^{n-1}}{z^n} + \frac{u^n}{z^n(z-u)}$$

and (3.1) it follows that for each integer n

$$H(z) = \int \frac{u^n}{z^n} \frac{1}{\Phi(u)} \frac{d\sigma(u)}{z-u},$$

so that for each polynomial p

$$p(z)H(z) = \int \frac{p(u)}{\Phi(u)} \frac{d\sigma}{z-u}.$$

Consider the polynomials for which $|p| < \Phi$. Then the total variation of $\int_{-\infty}^{u}(p(v)/\Phi(v))d\sigma$ is at most that of σ. This enables us to invoke Lemma 2.3 with $H(z)$ replaced by $p(z)H(z)$ and conclude that

$$\int \frac{\log^+ |p(x+iy)H(x+iy)|}{1+x^2} dx < A, \quad 0 < y \leqq y_0. \tag{3.3}$$

This and (2.4) (with H defined by (3.2)) enable us to obtain

$$\int \frac{\log^+ |p(x+iy)|}{1+x^2}\,dx < A + C, \qquad 0 < y \leqq \min(y_0, y_1),$$

whence by Fatou's lemma

$$\int \frac{\log^+ |p(x)|}{1+x^2}\,dx < A + C.$$

Since $A+C$ is independent of p this contradicts hypothesis (1) and completes the proof.

4. The semi-infinite interval. Suppose that Φ is a continuous function defined for $x \geqq 0$ and satisfying the conditions

(i) $$\Phi(x) > 0, \qquad 0 \leqq x < \infty;$$

(ii) $$\lim_{x\to\infty} x^n/\Phi(x) = 0, \qquad n = 0, 1, \cdots.$$

We ask when the set of functions $x^n/\Phi(x)$, $n=0, 1, \cdots$, is fundamental in the space C_0^+ of functions continuous for $x \geqq 0$, vanishing at ∞, and normed by the maximum. The answer can be deduced immediately from Theorem 1 and the following one.

THEOREM 2. *The set of functions $x^n/\Phi(x)$, $n=0, 1, \cdots$, is fundamental in C_0^+ if and only if the set $x^n/\Phi(x^2)$, $n=0, 1, \cdots$, is fundamental in C_0.*

First suppose that $x^n/\Phi(x^2)$ is fundamental in C_0. Let ϕ belong to C_0^+ and let ϵ be an arbitrary positive number. By hypothesis there exists a polynomial p such that

$$\left| \frac{p(x)}{\Phi(x^2)} - \phi(x^2) \right| < \frac{\epsilon}{2}, \qquad -\infty < x < \infty.$$

Replace x by $-x$ and add the two inequalities. This leads to

$$\left| \frac{p(x) + p(-x)}{\Phi(x^2)} - \phi(x^2) \right| < \epsilon, \qquad -\infty < x < \infty.$$

Now $p(x)+p(-x)$ is an even polynomial, say $q(x^2)$. Then

$$\left| \frac{q(x)}{\Phi(x)} - \phi(x) \right| < \epsilon, \qquad 0 \leqq x < \infty.$$

The converse is less trivial. Suppose that $x^n/\Phi(x)$ is fundamental n C_0^+. The problem is to show that the equations

$$\int \frac{x^n}{\Phi(x^2)} d\sigma(x) = 0, \qquad n = 0, 1, \cdots, \tag{4.1}$$

have only the trivial solution σ. There is no harm in supposing that σ is normalized: $\sigma(0)=0$, $\sigma(x)=[\sigma(x+)+\sigma(x-)]/2$.

First let $n=2m$ and make the change of variable $x^2=\zeta$ in (4.1). Then

$$\int_0^\infty \frac{\zeta^m}{\Phi(\zeta)} d[\sigma(\zeta^{1/2}) - \sigma(-\zeta^{1/2})] = 0, \quad m = 0, 1, \cdots.$$

Since $\zeta^m/\Phi(\zeta)$ is fundamental it follows that $\sigma(\zeta^{1/2})-\sigma(-\zeta^{1/2})$ is substantially a constant. By the normalization the constant must be zero, so that σ is even.

On the other hand, if $n=2m+1$, $m=0, 1, \cdots$, the equations (4.1) become (since σ is even)

$$\int_0^\infty \frac{x^{2m+1}}{\Phi(x^2)} d\sigma(x) = 0, \qquad m = 0, 1, \cdots.$$

Let $x^2=\zeta$ once again and this in turn becomes

$$\int_0^\infty \frac{\zeta^m}{\Phi(\zeta)} \zeta^{1/2} d\sigma(\zeta^{1/2}) = 0.$$

Hence

$$\int_0^x \zeta^{1/2} d\sigma(\zeta^{1/2}) = 0, \qquad x \geqq 0,$$

or

$$\int_0^y u\,d\sigma(u) = 0, \qquad y \geqq 0.$$

Integrate by parts to obtain

$$y\sigma(y) = \int_0^y \sigma(u)du,$$

so that $\sigma(y)$ is constant for $y \geqq 0$. Being even it is constant for $-\infty < y < \infty$.

5. **The problem for $L^p(-\infty, \infty)$.** Suppose $p \geqq 1$ and let $x^n k(x) \in L^p(-\infty, \infty)$, $n=0, 1, \cdots$. The problem now is to determine when $x^n k(x)$, $n=0, 1, \cdots$, is fundamental in L^p. By the arguments used in the introduction we may suppose that $k(x)=1/\Phi(x)$, where

(i) $\Phi(x) > 0$ *almost everywhere in* $(-\infty, \infty)$;

(ii) $x^n/\Phi(x) \in L^p$, $n = 0, 1, \cdots$,

and confine our attention to *real* L^p space.

THEOREM 3. *In order that the set of functions* $x^n/\Phi(x)$, $n=0, 1, \cdots$, *subject to conditions* (i) *and* (ii) *be fundamental in* L^p *it is necessary and sufficient that* (1) *hold, where the upper bound is taken over all real polynomials* p *such that* $\|p/\Phi\| < 1$.

The norm is of course

$$\|f\| = \left(\int |f(x)|^p dx\right)^{1/p}.$$

Suppose that $x^n/\Phi(x)$ is fundamental. Let $k = \|(1+x^2)^{-1}\|$ and $c_n = (1/2k)(1-1/n)$. For each n there exists a polynomial p_n such that

$$\left\|\frac{p_n(x)}{\Phi(x)} - \frac{c_n}{1+x^2}\right\| < \frac{1}{n}.$$

Then

$$\left\|\frac{p_n(x)}{\Phi(x)}\right\| < \frac{1}{n} + c_n\|(1+x^2)^{-1}\| = \frac{1}{n} + \frac{1}{2k}\left(1 - \frac{1}{n}\right)k$$

$$= \frac{1}{2}\left(1 - \frac{1}{n}\right) < 1, \qquad n = 1, 2, \cdots.$$

Furthermore

$$\lim_{n\to\infty}\left\|\frac{p_n(x)}{\Phi(x)} - \frac{c_n}{1+x^2}\right\| = 0,$$

so that for a subsequence of the p_n, which we continue to denote by p_n,

$$\lim_{n\to\infty}\frac{p_n(x)}{\Phi(x)} = \frac{1}{2k}\,\frac{1}{1+x^2} \qquad \text{a.e.}$$

Hence if (1) is false we have, by Fatou's lemma, that (1.5) holds. Because of condition (ii), $x^n/\Phi(x) \in L$ for each n. Therefore, by Lemma 1.1 with $a(x) = 1/\Phi(x)$ there exist a function $b(x)$ in every L^q, in particular in $L^{p'}$ such that

$$\int_{-\infty}^{\infty}\frac{x^n}{\Phi(x)}\,b(x)dx = 0, \qquad n = 0, 1, \cdots.$$

This contradicts the assumption that $x^n/\Phi(x)$ is fundamental, so that (1) cannot be false.

As for the sufficiency, suppose (1) holds over the prescribed range of polynomials. If $x^n/\Phi(x)$ is not fundamental there exists a function $\phi(x)$ in $L^{p'}$ such that $\phi \not\equiv 0$ and

$$\int \frac{u^n}{\Phi(x)} \phi(x) dx = 0, \qquad n = 0, 1, 2, \cdots.$$

We may suppose

$$\left(\int \left| \phi(x) \right|^{p'} dx \right)^{1/p'} \leqq 1.$$

Now define

$$H(z) = \int \frac{\phi(u)}{\Phi(u)} \frac{du}{z - u}.$$

As in §3 we may conclude that

$$p(z)H(z) = \int \frac{p(u)}{\Phi(u)} \frac{\phi(u)du}{z - u}.$$

Suppose that we restrict ourselves to polynomials p such that $\|p/\Phi\| < 1$. Then the total variation of

$$\sigma(u) = \int_{-\infty}^{u} \frac{p(v)}{\Phi(v)} \phi(v) dv$$

is at most 1, by Hölder's inequality. Consequently by Lemma 2.2 once again the inequality (3.3) is valid. The rest of the proof proceeds as in §3.

I have not found an analogue for Theorem 2 of sufficient simplicity to record here.

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Cornell University