NONCOMMUTATIVE JORDAN ALGEBRAS OF CHARACTERISTIC 0

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Jordan algebras are commutative algebras satisfying the identity

$$(x^2a)x = x^2(ax). \tag{1}$$

These algebras have been studied extensively.

A natural generalization to noncommutative algebras is the class of algebras A satisfying (1). Linearization of (1), if the base field contains at least 3 elements, yields

$$(xy + yx, a, z) + (yz + zy, a, x) + (zx + xz, a, y) = 0 \tag{2}$$

where (x, y, z) denotes the *associator* $(x, y, z) = (xy)z - x(yz)$. If A contains a unity element 1, and if the characteristic is $\neq 2$, then $z = 1$ in (2) implies

$$(y, a, x) + (x, a, y) = 0, \tag{3}$$

or, equivalently,

$$(xa)x = x(ax). \tag{4}$$

That is, A is *flexible* (a weaker condition than commutativity). If a unity element is adjoined to A in the usual fashion, then a necessary and sufficient condition that (2) be satisfied in the extended algebra is that both (2) and (3) be satisfied in A.

We define a *noncommutative Jordan algebra A* over an arbitrary field F to be an algebra satisfying (1) and (4). These algebras include the best-known nonassociative algebras (Jordan, alternative, quasi-associative, and—trivially—Lie algebras). In 1948 they were studied briefly by A. A. Albert in [1, pp. 574–575],[2] but the assumptions (1) and (4) seemed to him inadequate to yield a satisfactory theory, and he restricted his attention to a less general class of algebras which he called "standard." In this paper, using Albert's method of trace-admissibility[3] and his results for trace-admissible algebras, we give a

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[2] Numbers in brackets refer to the references cited at the end of the paper.

[3] We use the formulation given in [2] as being more convenient for characteristic 0 than the modified version presented in [4]. We are limited to the characteristic 0

structure theory for noncommutative Jordan algebras of characteristic 0.

1. Some elementary properties. It is shown in [**1**] that flexibility implies that (1) is equivalent to any one of the following:

$$x^2(xa) = x(x^2a),\ (xa)x^2 = x(ax^2),\ (ax)x^2 = (ax^2)x, \tag{5}$$

and that (1) and (5) imply that A of characteristic $\neq 2$ is *Jordan-admissible*; that is, the commutative algebra A^+ in which multiplication is defined by $x \cdot y = (xy+yx)/2$ is a Jordan algebra.

It seems not to have been noted that every flexible Jordan-admissible algebra of characteristic $\neq 2$ satisfies (1), so that A is a noncommutative Jordan algebra if and only if A is flexible and Jordan-admissible. For, denoting right and left multiplications in A by R_x and L_x respectively, $(x^2 \cdot a) \cdot x = x^2 \cdot (a \cdot x)$ implies

$$[R_{x^2} + L_{x^2},\ R_x + L_x] = 0. \tag{6}$$

But (4) implies $[L_x, R_x]=0$, while (3) implies both $[L_y, R_x] = [R_y, L_x]$ and

$$R_{xy} - R_xR_y = L_{yx} - L_xL_y. \tag{7}$$

In particular, we have $[L_{x^2}, R_x] = [R_{x^2}, L_x]$, $R_{x^2} - R_x^2 = L_{x^2} - L_x^2$. Then (6) gives $0 = [R_{x^2}+L_{x^2},\ R_x] + [R_{x^2}+L_{x^2},\ L_x] = [2L_x^2 - L_{x^2} + R_x^2,\ R_x] + [2R_{x^2} - R_x^2 + L_x^2,\ L_x] = 2[L_{x^2}, R_x] + 2[R_{x^2}, L_x] = 4[L_{x^2}, R_x]$, implying (1). It is not possible to derive (4) from Jordan-admissibility even for algebras containing a unity element, for there are known examples of Jordan-admissible algebras which are not flexible, but which do contain 1 [**3**, p. 186].

Any noncommutative Jordan algebra of characteristic $\neq 2$ is power-associative.[4] For, defining $x^{k+1} = x^kx$, we may prove $x^\lambda x^\mu = x^{\lambda+\mu}$ by induction on $\lambda+\mu = n$. Since this is true for $n=2, 3$ by (4), we assume $x^\lambda x^\mu = x^{\lambda+\mu}$ for all $\lambda+\mu < n$, $n \geqq 4$. Let $a = x^{n-3}$ in (5): $xx^{n-1} = x^{n-2}x^2 = x^{n-1}x = x^n$. We need to prove $x^{n-\alpha}x^\alpha = x^n$ for $\alpha = 1, \cdots, n-1$, and prove this by proving $x^{n-\alpha}x^\alpha = x^n = x^\alpha x^{n-\alpha}$ by induction on α. This has been proved for $\alpha = 1$, and we assume $x^{n-\beta}x^\beta = x^n = x^\beta x^{n-\beta}$ for all $\beta \leqq \alpha < n-1$, and prove $x^{n-(\alpha+1)}x^{\alpha+1} = x^n = x^{\alpha+1}x^{n-(\alpha+1)}$ as follows. Let

case by the fact that, for an absolutely primitive idempotent u in a general (commutative) Jordan algebra A of characteristic $p>0$, the structure of $A_u(1)$—the subalgebra on which u acts as an identity—is not known. When this result is known, it may yield not only a determination of (commutative) Jordan algebras of degree one, but also, by Albert's refinement in [**4**] of the trace-admissibility technique, a structure theory for noncommutative Jordan algebras of characteristic $p>0$.

[4] This is proved for characteristic $\neq 2, 3, 5$ in [**1**, p. 574].

$a=x^{n-(\alpha+2)}$, $y=x$, $z=x^\alpha$ in (2). Since the sum of any three or fewer exponents is $<n$, we have $0=2(x^2, a, x^\alpha)+4(x^{\alpha+1}, a, x)=2x^{n-\alpha}x^\alpha-2x^2x^{n-2}+4x^{n-1}x-4x^{\alpha+1}x^{n-(\alpha+1)}=4x^n-4x^{\alpha+1}x^{n-(\alpha+1)}$. But (3) implies $0=2(x^\alpha, a, x^2)+4(x, a, x^{\alpha+1})=4x^n-4x^{n-(\alpha+1)}x^{\alpha+1}$.

2. **Trace-admissibility.** A bilinear function $\tau(x, y)$ on a power-associative algebra A and with values in the base field F is called an *admissible trace function* for A (and A is called *trace-admissible*) in case

(i) $\tau(x, y)=\tau(y, x)$,
(ii) $\tau(xy, z)=\tau(x, yz)$,
(iii) $\tau(e, e)\neq 0$ for any idempotent e of A,
(iv) $\tau(x, y)=0$ if xy is nilpotent.

If A contains 1, then the use of a linear function $\delta(x)=\tau(1, x)$ may be substituted for that of $\tau(x, y)$, and it is known [2, p. 319] that (i)–(iv) are equivalent to:

(I) $\delta(xy)=\delta(yx)$,
(II) $\delta((xy)z)=\delta(x(yz))$,
(III) $\delta(e)\neq 0$ for any idempotent e of A,
(IV) $\delta(x)=0$ if x is nilpotent.

THEOREM. *Any noncommutative Jordan algebra A of characteristic* 0 *is trace-admissible.*

PROOF. Since any subalgebra of a trace-admissible algebra is trace-admissible, it is sufficient to prove this theorem under the assumption that A has a unity element 1. It is well known that, since the characteristic is 0,

$$\delta(x) = \text{Trace } R_x^+ = (1/2) \text{ Trace } (R_x + L_x) \tag{8}$$

is an admissible trace function for the Jordan algebra A^+. We shall show that $\delta(x)$ in (8) is also an admissible trace function for A.

Since powers in A and A^+ coincide, (III) and (IV) are valid in A because they hold in A^+. Flexibility implies (I), for $\delta(xy)-\delta(yx)=(1/2)$ Trace $(R_{xy}+L_{xy}-R_{yx}-L_{yx})=(1/2)$ Trace $(R_xR_y-R_yR_x+L_yL_x-L_xL_y)=0$ by (7). Now (II) in A^+ implies

$$\delta((z\cdot y)\cdot x) = \delta(z\cdot(y\cdot x)). \tag{9}$$

Hence (3), (I), and (9) yield $4\delta((xy)z)=\delta(2(xy)z+2(xy)z)=\delta(2x(yz)-2(zy)x+2z(yx)+2(xy)z)=\delta(4x(yz)-x(yz)-(yz)x-(zy)x-x(zy)+z(yx)+(yx)z+(xy)z+z(xy))=4\delta(x(yz))-4\delta((z\cdot y)\cdot x)+4\delta(z\cdot(y\cdot x))=4\delta(x(yz))$, implying (II).

3. **Structure theory.** Because of the general results in [2], the structure theory for noncommutative Jordan algebras of characteristic 0

may be given as a set of corollaries to our theorem in §2. Define the *radical* N of A to be the maximal nilideal[5] of A, and A to be *semi-simple* if its radical is 0. Define A to be *simple* if A is simple in the ordinary sense and not a nilalgebra.

COROLLARY 1. *The radical N of any noncommutative Jordan algebra A of characteristic 0 is the set of all z satisfying $\delta(xz)=0$ for every x in A, and coincides with the radical of the Jordan algebra A^+. Also A/N is semisimple.*

COROLLARY 2. *Any semisimple noncommutative Jordan algebra A of characteristic 0 is uniquely expressible as a direct sum $A=A_1 \oplus \cdots \oplus A_t$ of simple ideals A_i.*

COROLLARY 3. *The simple noncommutative Jordan algebras of characteristic 0 are*

(a) *the simple (commutative) Jordan algebras,*
(b) *the simple flexible algebras of degree two,*
(c) *the simple quasiassociative algebras.*

The simple quasiassociative algebras are determined in [1, Chap. V] and are further studied in [5]. A set of necessary and sufficient conditions for the multiplication table of any simple flexible algebra of degree two is given in [1, p. 588] but, inasmuch as these algebras include such interesting examples as the Cayley-Dickson algebras and the 2^t-dimensional algebras obtained by the Cayley-Dickson process [6], a complete determination of those which are not commutative remains an interesting problem.

REFERENCES

1. A. A. Albert, *Power-associative rings*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 552–593.

2. ———, *A theory of trace-admissible algebras*, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 317–322.

3. ———, *New simple power-associative algebras*, Summa Brasiliensis Mathematicae vol. 2 (1951) pp. 183–194.

4. ———, *The structure of right alternative algebras*, Ann. of Math. vol. 59 (1954) pp. 408–417.

5. C. M. Price, *Jordan division algebras and the algebras $A(\lambda)$*, Trans. Amer. Math. Soc. vol. 70 (1951) pp. 291–300.

6. R. D. Schafer, *On the algebras formed by the Cayley-Dickson process*, Amer. J. Math. vol. 76 (1954) pp. 435–446.

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[5] One cannot hope to prove that this radical is nilpotent, or even solvable. For every Lie algebra is its own radical by this definition.