CONTINUOUS COLLECTIONS OF DECOMPOSABLE CONTINUA ON A SPHERICAL SURFACE

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In this paper a study is made of continuous collections of decomposable continua on a spherical surface. Properties of the decomposition spaces of such collections filling up continua are established, and a characterization of the decomposition spaces of such collections filling up a spherical surface is obtained.

The results of the present paper are related to certain results obtained by R. D. Anderson. He has shown [2] that there is a continuous collection of nondegenerate continua filling up the plane which is with respect to its elements homeomorphic to the plane, and that there is a continuous collection of continua filling up a plane one-dimensional continuous curve which is with respect to its elements homeomorphic to the plane. On the other hand, he has obtained [1] a characterization of the plane decomposition spaces of continuous collections of nondegenerate continuous curves filling up a plane continuum. Such decomposition spaces are special types of hereditary continuous curves. The results of the present paper show that the decomposition spaces of continuous collections of decomposable continua filling up compact plane continua are special types of hereditary continuous curves, and consequently that even under this weaker hypothesis such collections are dimension reducing if they fill up two-dimensional continua and are dimension preserving if they fill up one-dimensional continua. It might be noted that there do exist such collections filling up one-dimensional continua, but that Theorem 2 of this paper gives a strong restriction about the nature of such collections.

Theorems about continuous collections of continua in a compact metric space can be stated equivalently in terms of monotone interior transformations (p. 130 of reference [10]).

It is understood throughout this paper that space is compact and metric.

DEFINITIONS. If H is a point set and ϵ is a positive number, $V(\epsilon, H)$ denotes the set of all points P such that some point of H is at a distance from P of less than ϵ. If H and K are point sets, $d(H, K)$ will be used to denote the greatest lower bound of the set of all positive numbers ϵ such that K intersects $V(\epsilon, H)$. For two

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such sets, $S(H, K)$ is the greatest lower bound of the set of all positive numbers ϵ such that K lies in $V(\epsilon, H)$ and H lies in $V(\epsilon, K)$.

If the capitalized letter R denotes a collection of closed point sets, the corresponding capitalized German letter $\mathfrak{R}$ denotes the decomposition space in which the points are the elements of R and if p and q are points of $\mathfrak{R}$, the distance between p and q is $S(p, q)$. R^* denotes the sum of all the elements of R.

The following lemma is well known and no argument is given for it here:

LEMMA 1. *If R is a collection of closed and compact point sets in a completely separable, metric space T, then $\mathfrak{R}$ is a completely separable, metric space.*

DEFINITIONS. Let G denote a continuous collection of mutually exclusive continua.

If k is a subcontinuum of an element g of G and k is not a subset of the closure of $g-k$, then $m(k)$ denotes the least number such that no point of k is at a distance more than that number from $g-k$. If k is such a subcontinuum of an element g of G, ϵ is a positive number which is not greater than $m(k)$ and z is the set of all points of k at a distance from $g-k$ of not less than ϵ, $M(\epsilon, k)$ denotes the least upper bound of the set of all numbers e such that there exist a point P of z and sequences $g_1, g_2, \cdots, a_1, a_2, \cdots, b_1, b_2, \cdots$ and $e_1, e_2, \cdots$ such that, for each positive integer n, e_n is a positive number and a_n and b_n are points of the element g_n of G and each subcontinuum of g_n containing a_n and b_n contains a point at a distance from k of not less than e_n and the sequence $g_1, g_2, \cdots$ converges to g, the sequences $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ converge to P and $e_1, e_2, \cdots$ converges to e.

If ϵ is a positive number and g is an element of G containing a subcontinuum k for which $m(k)$ is not less than ϵ, let $C(\epsilon, g)$ denote the least upper bound of the set of all numbers $M(\epsilon, k)$ for all such subcontinua k of g. If, for each subcontinuum k of g, $m(k)$ is less than ϵ, let $C(\epsilon, g)$ be zero.

The function $C(\epsilon, g)$ furnishes a measure of the lack of equicontinuity of convergence of those sequences of elements of G which converge to g. The following lemma is proved in the first paragraph on p. 592 of reference [3]:

LEMMA 2. *If K is a continuous collection of mutually exclusive continua and $\mathfrak{K}$ is a compact metric continuum, there is a subcollection H of K which is a dense inner limiting set in $\mathfrak{K}$ such that if h is an element of H and e is a positive number, $C(e, h)$ is zero.*

J. H. Roberts obtained [7] the special case of this lemma in which all of the elements of the collection K are arcs.

DEFINITIONS. Throughout the remainder of this paper S will denote a spherical surface in three-dimensional Euclidean space and G will denote a continuous collection of mutually exclusive decomposable continua filling up a continuum on S. The metric used is the ordinary distance between points. If C is a circle on S which is not a great circle, the *center* of C is that point P of S which is equidistant from all points of C and is at a distance from C of less than $2^{1/2}\cdot r$, where r is the radius of S. The distance from P to C is the *radius* of C. If P is a point of S, I is a closed number interval of positive numbers less than $2^{1/2}\cdot r$, and e is a positive number which is not greater than $2r$, then $C(P, I, e)$ denotes the collection of all circles on S with radius in I and center a point of S at a distance from P of less than e. The *radius of a domain* D on S is the least number e such that if J is a circle on S, which is not a great circle, and the complementary domain of J containing its center is a subset of D, the radius of J is less than or equal to e.

The statement that the point P of continuum M is a *local separating point* of M means that there is a domain R with respect to M containing P such that if C is the component of R containing P, $R-P$ is the sum of two mutually separated sets each intersecting C (p. 61 of reference [**10**]).

THEOREM 1. *If G^* is a subset of the boundary of a connected open set lying in $S-G^*$, there is a subcollection R of G which is a dense inner limiting set in $\mathfrak{G}$ and each of whose elements is a local separating point of $\mathfrak{G}$.*

PROOF. This theorem is established by showing that there is a subcollection N of G such that for each element n of N, it is possible to define a notion of a side of n so that n has either one or two sides. It is then shown that, speaking roughly, those elements of G, lying near n, lie in only one side of n, and that if there are elements of G in each side of n near n, n is a local separating point of $\mathfrak{G}$. It is next shown that there is a subcollection R of N such that for each element r of R, there are elements of G in each side of r near r.

Since G^* does not intersect two complementary domains of any element of G, not more than countably many elements of G have two complementary domains. There is a subcollection N of G, no element of which cuts S, such that $\mathfrak{N}$ is a dense inner limiting set in $\mathfrak{G}$, and such that if n is an element of N and e is a positive number, $C(e, n)$ is zero. This follows immediately from Lemma 2.

Let n denote an element of N. Since n is decomposable, it is the sum of two of its proper subcontinua, n_1 and n_2. There is an arc α'', lying except for its end points in $S-n$, which intersects n_1 at a point P_1 not in n_2 and intersects n_2 at a point P_2 not in n_1. By Theorem 34, p. 203 of reference [5], there are just two complementary domains, D_1 and D_2, of $n+\alpha''$, and α'' is a subset of the boundary of each of them. Since $C(e, n)$ is zero for every positive number e, there is a positive number e such that if g is an element of G, $S(g, n)$ is less than e, and A, B, C, and D are points of g such that $d(A, P_1)$, $d(B, P_1)$, $d(C, P_2)$, and $d(D, P_2)$ are less than e, then there are subcontinua g_1 and g_2 of g, g_1 containing A and B and g_2 containing C and D, such that no point of g_1 is at a distance from P_2 of less than $d(P_2, n_1)/2$, and no point of g_2 is at a distance from P_1 of less than $d(P_1, n_2)/2$. Let E and F denote points of α'' between P_1 and P_2 such that the subarcs EP_1 and FP_2 of α'' have diameters less than e.

Since G^* is a subset of the boundary of a connected set lying in its complement, there do not exist a point of that complement and two elements of G each of which separates n from that point. Therefore, there is a positive number c such that if g is an element of G and $S(g, n)$ is less than c, g does not separate n from EF in S. Suppose there is an element g of G for which $S(g, n)$ is less than both $d(n, \text{arc } EF)/2$ and c, which has subcontinua g_1 and g_2 which respectively lie in $D_1+\alpha''$ and $D_2+\alpha''$ and are irreducible from EP_1 to FP_2. If g_1 and g_2 have a common point in EP_1, let β denote one such point. If they do not, let β' denote a subarc of EP_1 irreducible from g_1 to g_2. There is an arc β, irreducible from g_1 to g_2, having the same end points as β' and not intersecting α'', except in points of β', such that the arc EP_1 in which β is substituted for β' has diameter less than e, and such that if g has a point Q on an open segment of β, then Q is a limit point of subsets of g from both sides of β. Let α' denote the arc α'' in which β is substituted for β'. Similarly, define arcs (or a point) γ' and γ for FP_2 and substitute γ for γ' in α' to obtain the arc α. Let $\Delta=\beta+\gamma+g_1+g_2$. There exist an arc from n to g_1 not intersecting $\alpha+g_2$ and an arc from n to g_2 not intersecting $\alpha+g_1$. Let D denote the complementary domain of Δ containing EF, and AB denote the irreducible subarc of α from Δ to Δ containing EF. By Theorem 34, p. 203 of reference [5], n and EF lie in different complementary domains of Δ.

There is an arc η from a point of EF to a point of n which does not intersect g and is such that if Z is an irreducible subarc of η from β to β (or from γ to γ), both complementary domains of $\beta+Z$ (or $\gamma+Z$) intersect g, and any two successive components on η of $\eta-\eta\cdot(\beta+\gamma)$, such that the subset of η between them lies on β, abut on β from different sides, and similarly for γ. In the order from EF to n, either η intersects β before γ or γ before β. Suppose η intersects γ before β (the other case can be handled similarly). Let ζ denote the subarc of η from EF to γ, and ϵ denote the closure of the next component on η of $\eta-\eta\cdot(\beta+\gamma)$. Let ξ denote the component of $\eta\cdot\gamma$ intersecting ζ and ϵ. ξ is either a point or an arc. There are points T and Q of g on γ such that ξ is between them, and no point of g on γ is between them. Let V denote the first point of ξ from T to Q, and let U denote the first point of ξ from Q to T. No point of $\zeta+\epsilon$ not in ξ is between T and Q on γ. Since ϵ and ζ abut on γ from different sides, by Theorem 32, p. 201 of reference [5], TV and UQ abut on $\epsilon+\xi+\zeta$ from different sides. There is a subcontinuum h of g containing T and Q such that none of its points is at a distance from P_1 of less than $d(P_1, n_2)/2$. If η intersects β, there is a point R of η at a distance less than e from P_1 such that the subarc of η from R to EF does not intersect β, and such that there is an arc μ from R to n which does not intersect $h+FP_1$. Let η' denote η if it does not intersect β, or a subarc from EF to n of the subarc of η from EF to R plus the arc μ, if η does intersect β. TV and UQ abut on η' from different sides. By Theorem 34(2), p. 203 of reference [5], T and Q lie in different components of $S-(\eta'+n+FP_1)$. However, there is a continuum, h, containing them that does not intersect $\eta'+n+FP_1$. This is a contradiction.

If there were a sequence $g_1, g_2, \cdots$ of continua of G converging to n such that for each positive integer i, g_i contained two subcontinua, g_i^1 and g_i^2, g_i^1 lying in D_1 and g_i^2 lying in D_2, such that each of the sequences $g_1^1, g_2^1, \cdots$ and $g_1^2, g_2^2, \cdots$ converged to a subcontinuum of n containing P_1 and P_2, then since $C(e, n)$ is zero for every positive number e, there would be a continuum in G which has the characteristics of that in the previous impossible supposition.

Therefore, there is a positive number d such that if g is an element of G and $S(g, n)$ is less than d, and g_1 and g_2 are subcontinua of g, neither intersecting α and each containing a point at a distance from P_1 of less than d and a point at a distance from P_2 of less than d, then either g_1 and g_2 lie in D_1 or they lie in D_2. Every element g of G for which $S(g, n)$ is less than d contains a subcontinuum q, not intersecting α, which contains a point at a distance from P_1 of less than d and a point at a distance from P_2 of less than d. If q lies in D_1, g will be said to lie on the D_1 side of n; if q lies in D_2, g will be said to lie on the D_2 side of n. Let Z_1 denote the set of all elements z of G for which $S(z, n)$ is less than d, and which lie on the D_1 side of n; and let Z_2 denote the similar set for D_2. $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ are mutually separated in $\mathfrak{G}$, and if n is in both $\overline{\mathfrak{Z}_1}$ and $\overline{\mathfrak{Z}_2}$, n is a local separating point of $\mathfrak{G}$.

Suppose there is an uncountable subcollection M of N such that no element of M is a local separating point of $\mathfrak{G}$. For each element m of M, let m_1 and m_2 denote two proper subcontinua of m whose sum is m, α_m denote an arc lying except for its end points in $S-m$, P_1^m and P_2^m denote the two end points of α_m, P_1^m lying in m_1 but not m_2, and P_2^m lying in m_2 but not m_1, and D_1^m and D_2^m denote the complementary domains of $m+\alpha_m$. There is a positive number d_m such that if g is an element of G and $S(g, m)$ is less than d_m, g lies on either the D_1^m side of m or the D_2^m side of m. There is a positive number e_m, less than d_m, and one of the complementary domains of $m+\alpha_m$, say E_m, such that if g is an element of G and $S(g, m)$ is less than e_m, then g is on the non-E_m side of m. It can be shown by an indirect argument that for each element m of M, there is a positive number δ such that if I is a closed interval of positive numbers, each of which is less than δ, there is a positive number e such that if C_1 and C_2 are elements of $C(P_1^m, I, e)$ and $C(P_2^m, I, e)$, respectively, and g is an element of G for which $S(g, m)$ is less than e, then a complementary domain of Cl $(m+g+$ int. C_1+int. $C_2)$ lying in E_m has in its boundary a subset of α_m irreducible from C_1 to C_2 and a subcontinuum of m irreducible from C_1 to C_2. The following statement is implied by the fact that no element of M separates S; its form is for convenience in concluding the statement in the next paragraph. If m is an element of M and ϵ is a positive number, there is a positive number δ such that if I is a closed interval of positive numbers each of which is less than δ, there is a positive number e such that if C_1 and C_2 are elements of $C(P_1^m, I, e)$ and $C(P_2^m, I, e)$, respectively, and g is an element of G for which $S(g, m)$ is less than e, then if μ is the set of all subcontinua of m irreducible from C_1 to C_2, and γ is the set of all subcontinua of g irreducible from C_1 to C_2, and h and k are elements of $\mu+\gamma$, not more than one complementary domain of C_1+C_2+h+k has radius more than ϵ.

For each element m of M, there exist a closed number interval I_m and a positive number f_m, which is less than every number in I_m, such that if C_1 and C_2 are elements of $C(P_1^m, I_m, f_m)$ and $C(P_2^m, I_m, f_m)$, respectively, then

(1) α_m has only one subarc, β_m, irreducible from C_1 to C_2;

(2) there is a subcontinuum of m irreducible from C_1 to C_2 and a complementary domain D_m of $m+\beta_m+C_1+C_2$ having β_m and that subcontinuum on its boundary such that if g is an element of G for which $S(g, m)$ is less than f_m, then no subcontinuum of g which is irreducible from C_1 to C_2 intersects D_m; and

(3) if g is an element of G for which $S(g, m)$ is less than f_m, μ is the set of all subcontinua of m irreducible from C_1 to C_2, γ is the set

of all subcontinua of g irreducible from C_1 to C_2, and h and k are elements of $\mu+\gamma$, then not more than one complementary domain of $h+k+C_1+C_2$ has radius more than half of the radius of D_m.

By repeated alternate applications of Lemma 1 and the theorem that some point of each uncountable point set in a completely separable, metric space is a condensation point of that set, there exist a closed number interval I, a closed number interval J, the least number of which is 3/4's of the greatest, a positive number d, and an uncountable subcollection M' of M such that for each element m of M', I is a subinterval of I_m, d is less than f_m, and the radius of D_m is in J. There are three elements h, k, and m of M' such that if x, y, and z are h, k, and m, then $S(x, y)$, $d(P_1^x, P_1^y)$, and $d(P_2^x, P_2^y)$ are less than d. Let C_1 and C_2 denote circles with radii in I and centers P_1^h and P_2^h, respectively. C_i is in $C(P_i^x, I_x, f_x)$, where i is either 1 or 2 and x is either h, k, or m. Let X, Y, and Z denote the elements of μ_x, μ_y, and μ_z on the boundary of D_x, D_y, and D_z, respectively. Suppose that two of the arcs β_h, β_k, and β_m separate one of the continua X, Y, and Z from one of the other two in $S-(C_1+C_2+\text{int. } C_1+\text{int. } C_2)$. Then the sum of those two continua, together with C_1+C_2, has two complementary domains, each of radius greater than the least number in J, which is greater than one-half of the radius of each of the domains D_h, D_k, and D_m. This contradicts condition (3). Therefore, two of the continua X, Y, and Z separate the third from $\beta_h+\beta_k+\beta_m$ in $S-(C_1+C_2+\text{int. } C_1+\text{int. } C_2)$. But this contradicts condition (2) for one of the continua h, k, or m.

Since not more than countably many elements of N fail to be local separating points of $\mathfrak{G}$, if $\mathfrak{R}$ is the set of all points of $\mathfrak{N}$ which are local separating points of $\mathfrak{G}$, $\mathfrak{R}$ is a dense inner limiting set in $\mathfrak{G}$.

THEOREM 2. *If a and b are elements of G, $\mathfrak{H}$ is an irreducible subcontinuum of $\mathfrak{G}$ from a to b, and α is an arc in $(S-H^*)+a+b$ which is irreducible from a to b, then H^* is not a subset of the boundary of any complementary domain of $H^*+\alpha$ having α in its boundary.*

PROOF. Suppose this theorem is false. Then there exist elements a and b of G, a subcollection H of G, and an arc α as stated in the hypothesis of the theorem for which there is a complementary domain D of $H^*+\alpha$ having $H^*+\alpha$ as its boundary. By Theorem 1 of this paper and Corollary 9.21 of reference [10], there are two elements p and q of H such that $C(e, p)$ and $C(e, q)$ are zero for every positive number e, and such that $\mathfrak{H}-(p+q)$ is the sum of two mutually separated sets, one of which contains both a and b. Since $\mathfrak{H}$ is an irreducible continuum from a to b, p is a cut point of $\mathfrak{H}$.

The point set p is the sum of two of its proper subcontinua p_1

and p_2. Let A and B denote the end points of α on a and b, respectively. There exist a point P_1 of p_1 but not in p_2 and a point P_2 of p_2 but not in p_1. By an argument like that on p. 592 of reference [3], there are subcontinua h_{1A}, h_{1B}, h_{2A}, and h_{2B} of H^* such that h_{iX} is irreducible from P_i to X, where i is either 1 or 2 and X is either A or B, and such that P_1 is not in $h_{2A}+h_{2B}$ and P_2 is not in $h_{1A}+h_{1B}$; furthermore, h_{iA} does not have any point in common with $h_{1B}+h_{2B}$ not in p. There are two mutually exclusive circles J_1 and J_2 with centers P_1 and P_2, respectively, such that the interior of J_1 does not intersect $h_{2A}+h_{2B}$, and the interior of J_2 does not intersect $h_{1A}+h_{1B}$, and such that there are two mutually exclusive arcs γ_1 and γ_2 irreducible from P_1 and P_2, respectively, to an open segment of α, such that γ_1 lies, except for its points in $\alpha+\text{int. } J_1$, in D and does not intersect the interior of J_2, and γ_2 lies, except for its points in $\alpha+\text{int. } J_2$, in D and does not intersect the interior of J_1. Let C_1 and C_2 denote the end points of γ_1 and γ_2, respectively, on α.

Suppose A, B, C_1, and C_2 are in the order AC_1C_2B on α. (The other case can be treated similarly.) Since γ_1 and γ_2 abut on α from the same side, AC_1 and γ_2 abut on $\gamma_1+C_1C_2B$ from different sides. Thus A and the interior of J_2 lie in different complementary domains of $h_{1B}+\gamma_1+C_1C_2B$. But a subcontinuum of h_{2A} contains A, intersects the interior of J_2, and does not intersect $h_{1B}+\gamma_1+C_1C_2B$. This is a contradiction.

Theorem 3. *$\mathfrak{G}$ is a continuous curve.*

Proof. Suppose this theorem is false. Then there is a sequence $\mathfrak{H}_1, \mathfrak{H}_2, \cdots$ of mutually exclusive subcontinua of $\mathfrak{G}$ converging to a nondegenerate subcontinuum $\mathfrak{C}$ of $\mathfrak{G}$, no term of which intersects $\mathfrak{C}$. There is a proper subcontinuum $\mathfrak{D}$ of $\mathfrak{C}$ which is irreducible between two of its points, p and q, such that there exist a positive integer n and an arc α for which $H^*_{n+1}+H^*_{n+2}+\cdots$ intersects only one complementary domain of D^* and α lies, except for its end points, in that complementary domain and is irreducible from p to q. Since G is a continuous collection, each element of D is in the boundary of one of the complementary domains Q' and T' of $D^*+\alpha$ having α in their boundaries. Therefore, all of the elements of a subcontinuum $\mathfrak{E}$ of $\mathfrak{D}$, which is not a continuum of condensation in $\mathfrak{D}$, lie in the boundary of one of those complementary domains, say Q. Since $\mathfrak{D}$ is an irreducible continuum, the boundaries of Q' and T' intersect each element of D. Therefore, there is a dense set of elements of D each containing a point accessible from that one of the domains Q' and T' which is not Q, say T. Since $\mathfrak{E}$ is not a continuum of condensation in $\mathfrak{D}$, there is an arc β, lying in T except for its end points, which is irreducible between some two elements of E. Let $\mathfrak{B}$ denote an irreducible subcontinuum of $\mathfrak{E}$ between those two elements. That complementary domain of $B^*+\beta$ having β on its boundary and not intersecting α does not intersect Q. Therefore, B^* is a subset of the boundary of a complementary domain of $B^*+\beta$ having β in its boundary. This contradicts Theorem 2.

THEOREM 4. *Each subarc of $\mathfrak{G}$ contains uncountably many local separating points of $\mathfrak{G}$.*

PROOF. Suppose that a subarc $\mathfrak{H}$ of $\mathfrak{G}$ does not contain uncountably many local separating points of $\mathfrak{G}$. There is an element of H which is not a cut point of $\mathfrak{G}$ and which is not an end point of $\mathfrak{H}$. Therefore, there is an arc in $\mathfrak{G}$ having only two points in common with $\mathfrak{H}$. Let a and b denote those points of $\mathfrak{H}$, and let $\mathfrak{L}$ denote the subarc of $\mathfrak{H}$ from a to b. No element of L contains elements of L in two of its complementary domains. Let Z denote those elements g of G such that for each element of L, g is in the complementary domain of that element which contains the other elements of L. L^* is a subset of $\overline{Z}^*$. The argument may be concluded as in the proof of Theorem 3.

THEOREM 5. *Each nondegenerate subcontinuum of $\mathfrak{G}$ contains uncountably many local separating points of $\mathfrak{G}$.*

PROOF. Each nondegenerate subcontinuum of $\mathfrak{G}$ is a continuous curve and so contains an arc, which by Theorem 4 contains uncountably many local separating points of $\mathfrak{G}$. A list of characterizations of continua having this property is presented on p. 248 of reference [**4**].

THEOREM 6. *If G fills up S, $\mathfrak{G}$ is a dendron.*

PROOF. A contradiction to Theorem 2 follows immediately the supposition that $\mathfrak{G}$ contains a simple closed curve. This theorem and the next also follow easily from various results of this paper and theorems proved either by Moore [**6**], Vietoris [**8**], or Whyburn [**9**].

THEOREM 7. *If G fills up S and none of its elements is the boundary of three of its complementary domains, $\mathfrak{G}$ is an arc.*

PROOF. The decomposition space of a collection of elements of G filling up a complementary domain of an element of G is connected. Hence $\mathfrak{G}$ has no junction point and is an arc.

THEOREM 8. *For each dendron D, there is a continuous collection G of mutually exclusive decomposable continua filling up S such that $\mathfrak{G}$ is homeomorphic to D.*

PROOF. This result is an immediate corollary of an unpublished result announced by R. D. Anderson in the Proceedings of the International Congress (1950). An indication of an argument not dependent on this result is given here.

It is sufficient to show that there exists a continuous collection G' of mutually exclusive continua (not necessarily decomposable) filling up a simple closed curve J plus its interior I with each element of G' intersecting J and with $\mathfrak{G}'$ homeomorphic to D. Let $x_1, x_2, \cdots$ be a dense collection of cut points of D including all the emanation points of D. Proceed inductively. Let t_1 be a continuum in $J+I$ which is the boundary of each of its complementary domains in $J+I$ and whose complementary domains all intersect J and are in 1-1 correspondence T_1 with the complementary domains of x_1 in D. For each i, let t_i be a continuum in the complementary domain of $J+I-(t_1+t_2+\cdots+t_{i-1})$ corresponding under $T_1, \cdots, T_{i-1}$ to the complementary domain of $D-(x_1+\cdots+x_{i-1})$ containing x_i with t_i satisfying conditions analogous to those imposed on t_1. By employing rather obvious further conditions on t_i with respect to $t_1, \cdots, t_{i-1}$ it can be insured that $t_1, t_2, \cdots$ is a continuous collection admitting the definition of G' as required.

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