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DIFFERENTIAL IDEALS[1]

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0. **Introduction.** In this paper we investigate the membership of power products in certain differential ideals. The questions examined were motivated by results by Levi,[2] which we use extensively. Levi has obtained for $[y^p]$ and $[uv]$ sufficiency conditions for membership of a pp. in the ideal, which tests membership, in certain cases, by a calculation using only the weight and degree of the pp. In Theorem IV we show that a more refined criteria is required for the determination of membership of a pp. in $[y^p]$. Whether a necessity criteria for membership of a pp. in $[uv]$ will require more information than the weight and degree of the pp. is not known.

Levi has also shown that the totality of pp. in u and v are divided by a single calculation into three nonempty sets: the α-terms, which are outside the ideal $[uv]$, another set all of whose members are in the ideal, and a third set concerning whose elements membership in the ideal is undecided. The number of elements known to be outside the ideal is increased by Theorem II, and a dependence of the set of elements whose membership in $[uv]$ is undecided upon one of its proper subsets is demonstrated in Theorem III. Carrying out the reduction

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[1] The nomenclature and notation are as in J. F. Ritt's *Differential algebra*, Amer. Math. Soc. Colloquium Publications, vol. 33, New York, 1950.

[2] H. Levi, *On the structure of differential polynomials and on their theory of ideals*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 532–568.

process, obtained by Levi, for $[uv]$ will provide a positive answer for membership or nonmembership in the ideal of any particular pp. This process, usually requiring many computations to determine certain critical coefficients can be simplified by the results in our first section. It is shown that some of the coefficients can be predicted through use of the derivative subscripts alone, without carrying out the steps of the reduction. We also obtain, and in the last section utilize, a similar simplification of the reduction process for $[y^p]$.

1. **Invariance theorems.** A basic procedure in Levi's reduction process is the attainment of a congruence relation (1) $P \equiv -\sum c_i Q_i$ where the c_i are quotients of binomial coefficients. P and Q_i are either pp. in the y_j and the congruence is modulo $[y^p]$ or they are pp. in u and v and the congruence is modulo $[uv]$. We give a brief description of how this congruence relation is obtained for the case $[uv]$.

If $P = u_i v_j R$, where R is a pp. in u and v, we use $(uv)_{i+j} = \sum_{\alpha+\beta=i+j} d_{\alpha,\beta} u_\alpha v_\beta$ where $d_{\alpha,\beta} = C_\alpha^{\alpha+\beta}$ to obtain $0 \equiv R \sum d_{\alpha,\beta}\, u_\alpha v_\beta$ (modulo $[uv]$). ($C_\alpha^{\alpha+\beta}$ is the binomial coefficient.) Thus $Ru_i v_j \equiv -R(\sum_{\alpha+\beta=i+j;\alpha\neq i} c_{\alpha,\beta} u_\alpha v_\beta)$ (modulo $[uv]$) where $c_{\alpha,\beta} = C_\alpha^{\alpha+\beta}/C_i^{i+j}$. It is the last congruence which is used in the reduction process.

Before presenting statements and proofs of the theorems, we introduce and define the pertinent terms. The transition from P to $-c_i Q_i$, any term on the right of the congruence (1), is called the step which starts at P and ends at Q_i. The step which starts at P and ends at Q_i is also called a path of length one—a path which starts at P and ends at Q_i. Assume paths of length k have been defined. Take a path of length k which starts at P and ends at S, with the coefficient of S being c. There exists a congruence relation, obtained as described above: $cS \equiv -\sum c_i' T_i$. The transition from P to $-c_i' T_i$, any one of the terms of the congruence, is called a path of length $k+1$—a path which starts at P and ends at T_i. The concepts of step and path for $[y^p]$ and $[u_1 u_2 \cdots u_p]$ are defined analogously. We first turn our attention to the latter ideal.

Let $R = u_{1,i_1} u_{2,i_2} \cdots u_{p,i_p} U$ and $T = u_{1,j_1} u_{2,j_2} \cdots u_{p,j_p} U$, where U is a pp. in $u_1, u_2, \cdots, u_p$ and $\sum_{\alpha=1}^p i_\alpha = \sum_{\alpha=1}^p j_\alpha$. The first multiplier of the step which starts at R and ends at T is:

$$m(R, T) = \prod_{\alpha=1}^{p} (i_\alpha!) \Big/ \prod_{\alpha=1}^{p} (j_\alpha!).$$

The second multiplier of the same step is $M(R, T) = -1$. The first (second) multiplier of a path is the product of the first (second) multipliers of the steps which compose the path. The relation between multipliers and the coefficients arising in steps and paths is given in the following lemma.

LEMMA I. *If there is a path which starts at R and ends at T, the coefficient of T, at the end of the path, is $m(R, T)\ M(R, T)$.*

PROOF. Let R and T be the pp. defined above and let $n=\sum_{\alpha=1}^{p} i_\alpha$. Since $(u_1u_2u_3\cdots u_p)_n=cu_{1,i_1}u_{2,i_2}\cdots u_{p,i_p}+c'u_{1,j_1}\cdots u_{p,j_p}+$other terms, where

$$c = C_{i_p}^{d_p}C_{i_{p-1}}^{d_{p-1}}\cdots C_{i_2}^{d_2} = \frac{(d_p)!}{(i_1!)\cdots(i_p!)},$$

$$c' = C_{j_p}^{d'_p}\cdots C_{j_2}^{d'_2} = \frac{(d'_p)!}{(j_1!)\cdots(j_p!)},$$

$$d_k = \sum_{\alpha=1}^{k} i_\alpha, \qquad d'_k = \sum_{\alpha=1}^{k} j_\alpha$$

(note $d_p=d'_p=n$), the coefficient of T after making the step is seen to be:

$$-c'/c=-((i_1!)\cdots(i_p!))/((j_1!)\cdots(j_p!))=m(R,\ T)M(R,\ T).$$

An induction yields the corresponding result for paths of any length.

THEOREM I. *$m(R, T)$ is independent of the path.*

Let $R=u_{1,0}^{i_{1,0}}u_{1,1}^{i_{1,1}}\cdots u_{1,n}^{i_{1,n}}u_{2,0}^{i_{2,0}}u_{2,1}^{i_{2,1}}\cdots u_{2,n}^{i_{2,n}}\cdots u_{p,0}^{i_{p,0}}\cdots u_{p,n}^{i_{p,n}}$ and $T=u_{1,0}^{j_{1,0}}\cdots u_{1,n}^{j_{1,n}}\cdots u_{p,n-1}^{j_{p,n-1}}u_{p,n}^{j_{p,n}}$ where the total weights are the same, and for each i, the degree of R in u_i is the same as the degree of T in u_i. It will be shown that:

$$m(R, T) = \prod_{\alpha=1}^{n}\left\{(\alpha!)^{\sum_{\beta=1}^{p} i_{\beta,\alpha}}\right\}\Big/\prod_{\alpha=1}^{n}\left\{(\alpha!)^{\sum_{\beta=1}^{p} j_{\beta,\alpha}}\right\}. \tag{1}$$

The proof employs induction on the length of the path from R to T. Lemma I provides the proof for paths of length one. Assume (1) is correct for all paths of length less than t, and take any path starting at R and ending at T of length t. For the proper choice of $k_1,\cdots,k_p$ and $m_1,\cdots,m_p$, the first step of the path yields

$$Q = u_{1,0}^{i_{1,0}}\cdots u_{1,k_1}^{i_{1,k_1}-1}u_{1,k_1+1}^{i_{1,k_1}+1}\cdots u_{2,k_2}^{i_{2,k_2}-1}u_{2,k_2+1}^{i_{2,k_2}+1}\cdots u_{p,k_p}^{i_{p,k_p}-1}u_{p,k_p+1}^{i_{p,k_p}+1}\cdots u_{p,n}^{i_{p,n}}u_{1,m_1}u_{2,m_2}\cdots u_{p,m_p}$$

for which $m(R, Q)=\prod_\beta^p k_\beta=\prod_\beta^p m_\beta$. The path from Q to T contains only $t-1$ steps, and (1) is assumed valid for such paths.

$$m(Q, T) = \left[\prod_{\alpha=1}^{n} \{(\alpha!)^{\sum_\beta^p i_{\beta,\alpha}}\} \Big/ \prod_{\alpha=1}^{n} \{(\alpha!)^{\sum_\beta^p j_{\beta,\alpha}}\} \right] \prod_{\beta=1}^{p} (m_\beta!) \Big/ \prod_{\beta=1}^{p} (k_\beta!).$$

For $\beta = 1, \cdots, p$ the exponent in Q of u_{β,k_β} is $i_{\beta,k_\beta}-1$, not i_{β,k_β}, and the exponent of u_{β,m_β} is $i_{\beta,m_\beta}+1$, not i_{β,m_β}; thus in the numerator of the first factor of $m(Q, T)$, the exponent of each of $(k_1!), (k_2!), \cdots, (k_p!)$ is one too large and the exponent of $(m_1!), (m_2!), \cdots, (m_p!)$ is one too small. From the definition of the first multiplier of a path:

$$m(R, T) = m(R, Q)m(Q, T)$$

$$= \frac{\prod_{\beta=1}^{p} (k_\beta!)}{\prod_{\beta=1}^{p} (m_\beta!)} \left[\prod_{\alpha=1}^{n} \{(\alpha!)^{\sum_\beta^p i_{\beta,\alpha}}\} \Big/ \prod_{\alpha=1}^{n} \{(\alpha!)^{\sum_\beta^p j_{\beta,\alpha}}\} \right] \frac{\prod_{\beta}^{p} (m_\beta!)}{\prod_{\beta}^{p} (k_\beta!)}.$$

The proof of the lemma has been completed.

We turn to $[y^p]$ and establish the corresponding results concerning the invariance of the first multiplier for paths with fixed end points. Let $R = y_{i_1}^{a_1} y_{i_2}^{a_2} \cdots y_{i_n}^{a_n} Y$ and $T = y_{j_1}^{b_1} \cdots y_{j_n}^{b_n} Y$ where Y is a pp. in the y_i, $p = \sum_\alpha^n a_\alpha = \sum_\alpha^n b_\alpha$, no two of the i_k are the same and no two of the j_k are the same. The first multiplier of the step which starts at R and ends at T is: $m(R, T) = [(i_1!)^{a_1} \cdots (i_n!)^{a_n}]/[(j_1!)^{b_1} \cdots (j_n!)^{b_n}]$. Note that this is what is obtained from the first multiplier as defined for $u_1 u_2 \cdots u_p$ if no distinction is made among the u's. The second multiplier of the step which starts at R and ends at T is: $M(R, T) = -\prod_\alpha^n (a_\alpha!)/\prod_\alpha^n (b_\alpha!)$.

The first (second) multipliers of a path is the product of the first (second) multipliers of the steps which compose the path. The relationship between multipliers and coefficients arising in steps and paths is given in the following lemma.

LEMMA I′. *If there is a path which starts at R and ends at T, the coefficient of T at the end of the path is $m(R, T)M(R, T)$.*

PROOF. Let R and T be the pp. defined above, with $f = \sum_{\alpha=1}^n a_\alpha i_\alpha = \sum_{\alpha=1}^n b_\alpha j_\alpha$. In the expansion $(y^p)_f = c y_{i_1}^{a_1} \cdots y_{i_n}^{a_n} + c' y_{j_1}^{b_1} \cdots y_{j_n}^{b_n}$ + other terms, y^p can be considered a specialization of $u_1 u_2 \cdots u_p$ in which all the u's are the same. The coefficients c and c' are the coefficients of any one of the corresponding terms in $u_1, \cdots, u_p$ multiplied by a suitable binomial coefficient. Thus:

$$c = \frac{f!}{(i_1!)^{a_1} \cdots (i_n!)^{a_n}} C_{a_n}^{d_n} C_{a_{n-1}}^{d_{n-1}} \cdots C_{a_2}^{d_2}$$

and

$$c' = \frac{f!}{(j_1!)^{b_1} \cdots (j_n!)^{b_n}} C_{b_n}^{d'_n} \cdots C_{b_2}^{d'_2}$$

where $d_k = \sum_{\alpha=1}^{k} a_\alpha$ and $d'_k = \sum_{\alpha=1}^{k} b_\alpha$. (Note that $d_p = d'_p = n$.) It is easy to see that

$$c = \frac{f!}{\prod_{\alpha=1}^{n} (i_\alpha!)^{a_\alpha}} \cdot \frac{(d_n!)}{\prod_{\alpha=1}^{n} (a_\alpha!)} \quad \text{and} \quad c' = \frac{f!}{\prod_{\alpha=1}^{n} (j_\alpha!)^{b_\alpha}} \cdot \frac{(d'_n!)}{\prod_{\alpha=1}^{n} (b_\alpha!)} .$$

The coefficient of T after making the step is $-c'/c = m(R, T)M(R, T)$. An induction proof provides the corresponding result for paths of any length.

The second invariance theorem is analogous to Theorem I.

THEOREM I'. *$m(R, T)$ is independent of the path.*

Since the definition of the first multiplier for $[y^p]$ is obtained from the corresponding definition for $[u_1 \cdots u_p]$ by associating the u's, the proof of Theorem I' can be obtained from that for Theorem I by the same specialization.

2. **A dependence theorem.** Levi has shown that all pp. of the form $u_{i_1}u_{i_2} \cdots u_{i_n}v_{j_1} \cdots v_{j_t}$ with $n \leqq j_1 \leqq \cdots \leqq j_t$ are not in $[uv]$. A larger set of pp. whose nonmembership in $[uv]$ can be determined by a single calculation is obtained in the following theorem.

THEOREM II. *$P = u^s u_{i_1} u_{i_2} \cdots u_{i_n}(v_s)^r \notin [uv]$ if $r \leqq i_1 \leqq \cdots \leqq i_n$.*

It will be shown that $P \equiv a u^s u_{i_1-r} \cdots u_{i_n-r}(v_{s+n})^r + hT$ (mod $[uv]$) where a is a nonzero rational number and hT represents a linear combination of terms higher than the pp. $T = u^s u_{i_1-r} \cdots u_{i_n-r}(v_{s+n})^r$. Since T is an α-term, the fact, to be proved, that a is not zero will imply that $P \notin [uv]$. Throughout this section all congruences are modulo $[uv]$, and the subscripts $i_1, \cdots, i_n$ are assumed fixed and to satisfy $r \leqq i_1 \leqq \cdots \leqq i_n$.

LEMMA I. *$u^s u_{i_1-r} \cdots u_{i_j-r} v_k \equiv hT$ or 0 when $k < j+s$.*

PROOF. If $s=0$, the smallest value possible for j is 1, and $k=0$. We find $u_{i_1-r}v \equiv -\sum_{k=1}^{i_1-r} c_k u_{i_1-r-k} v_k$. This case is completed by observing that since $k > 0$, every term of the sum is higher than T.

If $s > 0$ and $j = 0$, the pp. contains $u^s v_k$ with $k < s$, and it is known that such pp. are in the ideal. We continue the proof, using induction on j. Assume the lemma true for all values less than j. By means of the reduction process:

$$u^s u_{i_1-r} \cdots u_{i_j-r} v_k$$
$$\equiv u^s u_{i_1-r} \cdots u_{i_{j-1}-r}\left(-\sum_{i=1}^{i_j-r} c_i\, u_{i_j-r-i} v_{k+i} - \sum_{i=1}^{k} c_i'\, u_{i_j-r+i} v_{k-i}\right).$$

Every term of the first sum is higher than T, and every term of the second, by the induction hypothesis, is congruent to zero or a linear combination of terms higher than T. The proof of the lemma is complete.

LEMMA II. *If* $i \geqq 0$, $u^s u_{i_1-r} \cdots u_{i_{j-1}-r} u_{i_j-r+q-i}(v_{s+j-1})^{q+1} \equiv hT$ *or* 0 *for* $q \geqq 0$.

PROOF. For $q=0$ and $i=0$, this is Lemma I. If $q=0$ and $i>0$ the pp. is higher than T. Assume the lemma valid for all values less than q (with any $i \geqq 0$).

$$u^s u_{i_1-r} \cdots u_{i_{j-1}-r} u_{i_j-r+q-i} v_{s+j-1} (v_{s+j-1})^q$$
$$\equiv u^s u_{i_1-r} \cdots u_{i_{j-1}-r} (v_{s+j-1})^s \Bigg[- \sum_{k=1}^{i_j-r+q-i} c_k u_{i_j-r+q-i-k} v_{s+j-1+k}$$
$$- \sum_{k=1}^{s+j-1} c_k'\, u_{i_j-r+q-i+k} v_{s+j-1-k} \Bigg].$$

Every term of the first sum, by induction, and every term of the second, by the preceding lemma, is congruent to zero or hT. The conclusion of the lemma is satisfied.

LEMMA III. *There is a nonzero rational number a for which*:

$$u^s u_{i_1-r} \cdots u_{i_{j-1}-r} u_{i_j-q} (v_{s+j-1})^{r-q} (v_{s+1})^q$$
$$\equiv a u^s u_{i_1-r} \cdots u_{i_{j-1}-r} u_{i_j-q-1} (v_{s+j-1})^{r-q-1} (v_{s+j})^{q+1} + hT.$$

PROOF.

$$u^s u_{i_1-r} \cdots u_{i_{j-1}-r} (u_{i_j} v_{s+j-1}) (v_{s+j-1})^{r-q-1} (v_{s+j})^q$$
$$\equiv u^s u_{i_1-r} \cdots u_{i_{j-1}-r} (v_{s+j-1})^{r-q-1} (v_{s+j})^q \Bigg[- \sum_{i=1}^{s+j-1} c_i u_{i_j-q+i} v_{s+j-i-1}$$
$$- \sum_{i=1}^{i_j-q} c_i'\, u_{i_j-q-i} v_{s+j-1+i} \Bigg].$$

By the first lemma, every term of the first sum is congruent to zero or hT. Every term of the second for which $i=2, 3, \cdots, i_j-q$ is, by the second lemma, congruent to hT or zero, and c_1' as the quotient of binomial coefficients is not zero. This gives us our result.

COROLLARY. *$u^s u_{i_1-r} \cdots u_{i_j-1} u_{i_j}(v_{s+j-1})^r \equiv a u^s u_{i_1-r} \cdots u_{i_j-r}(v_{s+j})^r + hT$, with $a \neq 0$.*

The corollary can be obtained by an r-fold application of the last lemma, and the proof of the theorem is completed through an n-fold iteration of the corollary.

3. **A dependence theorem.** The sufficiency condition which Levi obtains for membership in $[y^p]$ and $[uv]$ is that the weight must be small with respect to the degree. For $[uv]$ the weight must be less than $d_1 \cdot d_2$ where (d_1, d_2) is the signature of the pp. Thus for each pair (d_1, d_2) there exists a critical weight such that any pp. of signature (d_1, d_2) and weight less than the critical weight automatically lies in the ideal.

Let P be a pp. in u and v of signature (d_1, d_2). For all possible pairs (d_1', d_2'), where $1 \leqq d_1' \leqq d_1$ and $1 \leqq d_2' \leqq d_2$, we consider the weight of the factor of P of least weight and signature (d_1', d_2'), minus $d_1' \cdot d_2'$ (the critical weight). This set of numbers we call the weight sequence of P. If all the numbers of the weight sequence are non-negative, we say that P has a non-negative weight sequence. The weight of P minus $d_1 \cdot d_2$ is called the excess weight of P.

THEOREM III. *If no pp. with a non-negative weight sequence and excess weight of zero is in $[uv]$, then no pp. with a non-negative weight sequence is in $[uv]$.*

Let $w(P)$ = weight of P and $g(P) = d_1 \cdot d_2$, where (d_1, d_2) is the signature of P.

LEMMA I. *If $P = u_k Q$ and P is of signature (d_1, d_2), then $w(P) - g(P) = w(Q) - g(Q) + k - d_2$.*

PROOF. $w(Q) = w(P) - k$ and $g(Q) = (d_1 - 1) \cdot d_2 = g(P) - d_2$. Thus $w(P) - g(P) = w(Q) - g(Q) + k - d_2$.

Let the u_k-part of P be the pp. obtained by deleting from P all the v's, and all the u_j with $j \geqq k$, and let the v-part of P be the pp. obtained by deleting all the u's from P.

LEMMA II. *Given a pp. P (involving at least one v_j) with a non-negative weight sequence, there exists a pp. T such that:*

(i) *PT has a non-negative weight sequence.*

(ii) *There is a factor R of P with the v-part of R the same as the v-part of P, and the excess weight of RT equal to zero.*

PROOF. Let (d_1, d_2) denote the signature of P. Write $P = RU$, where U involves only $u_{d_2}, u_{d_2+1}, \cdots$ and no v's, and R, free of

$u_{d_2}, u_{d_2+1}, \cdots$, is of signature (d_1', d_2'); of course $d_2'=d_2$. We assume P involves a u_j with $j<d_2$ (i.e. $d_1'>0$) since it is known that if P lacks such a u_j, $P\notin[uv]$. $(u_{d_2-1})^{w(R)-g(R)}$ satisfies the conditions of T in the lemma. The second part of the conclusion will be demonstrated first, with the R and T already introduced.

$$\begin{aligned} w(RT) &= w(R) + (w(R) - g(R))(d_2 - 1) \\ &= d_2 w(R) - d_2 g(R) + g(R), \\ g(RT) &= (d_1' + w(R) - g(R))d_2 = d_1' \cdot d_2 + d_2 w(R) - d_2 g(R) \\ &= g(R) + d_2 w(R) - d_2 g(R). \end{aligned}$$

The second condition is satisfied.

We now show that PT has a non-negative weight sequence. Let S be any factor of PT with a minimum excess weight. The assumption that this excess weight is negative will produce a contradiction.

If the v-part of S were not the same as the v-part of PT, S would contain at most d_2-1 v's. After the deletion of all u_{d_2-1} from S, which the preceding lemma informs us will not increase the excess weight, a pp. S', which is a factor of P, is obtained. Since the excess weight of S' is non-negative (P has a non-negative weight sequence) the same must be true of S. Consequently, if the excess weight of S is to be negative, the v-part of S is the same as the v-part of PT.

The u_{d_2} part of S is identical with the u_{d_2} part of PT. Deletion of all u_j with $j\geqq d_2$ will not increase the excess weight. If S did not contain all the u_i in PT, with $i<d_2$, the addition of these u_i would decrease the excess weight. These deletions and additions, however, yield RT, which has an excess weight zero. Thus if S is to have a negative excess weight, the u_{d_2} part of S must be the same as the u_{d_2} part of PT. Similarly S cannot involve a u_j with $j>d_2$. S can only differ from RT by terms u_{d_2}, the addition or deletion of which do not affect the excess weight of the pp.

It has been shown that the minimum excess weight of any factor of PT is not negative, i.e. the weight sequence of PT is non-negative, and the proof of Lemma II is complete. We now return to the proof of the theorem.

Take any pp. P, of signature (d_1, d_2), with a non-negative weight sequence. There is a pp. T involving only u's such that PT has a non-negative weight sequence and a factor R of P such that RT has zero excess weight.

Let (r_1, r_2) be the signature of RT. There being but one α-term of weight $w(RT)$ and signature (r_1, r_2), namely $u^{r_1}(u_{r_1})^{r_2}$, $RT\equiv cu^{r_1}(u_{r_1})^{r_2}$ mod $[uv]$. The hypothesis of the theorem is the statement that $c\neq 0$.

$P=UR$, where U involves only $u_{d_2}, u_{d_2+1}, \cdots$; but as was seen in the previous section, the adjunction of such factors, as in U, to $u^{r_1}(u_{r_1})^{r_2}$ with $r_2=d_2$, does not enable the result to become a member of the ideal.

P after multiplication by T is not in the ideal, hence P cannot be a member. It may thus be sufficient to examine with reference to membership in the ideal those pp. which have a non-negative weight sequence and an excess weight of zero.

4. **Unusual pp. in $[y^p]$.** To demonstrate the need for a criteria for memberhip of a pp. in $[y^p]$, more refined than any test which consists of a single calculation in terms of the weight and degree of the pp., it will be sufficient to consider $[y^2]$. The manner in which this will be demonstrated may be stated more precisely following some conventions of notation. If $P=y_{i_1}\cdots y_{i_n}$ where the i_j are monotonically increasing, $(a_1, \cdots, a_n)$ is called the weight sequence of P and a_n the excess weight of P, where $a_j=(\sum_{k=1}^{j} i_k)-(j)(j-1)$. (We recall that for $[y^2]$ the critical weight function $w(2, d)=d(d-1)$.) Due to the monotony of the subscripts of the y's, $a_{i+1}-a_i \geqq a_i-a_{i-1}-2$ for $i=2, \cdots, n-1$. With these restrictions on the sequences, there is a one-to-one correspondence between weight sequences and pp. in y_j. In what follows, a pp. and its associated weight sequence are used interchangeably, and all congruences are modulo $[y^2]$.

The sufficiency test of Levi is that one of the entries of the weight sequence be negative. This condition is not necessary, as is shown by the following theorem.

THEOREM IV. *A pp. with a weight sequence consisting of* 0's, 1's *and* 2's *will be a member of* $[y^2]$ *if and only if somewhere in the sequence at least one of the following patterns appear*:

1, 2, 2, 1; 1, 2, 2, 2, 2, 0; 0, 2, 2, 2, 2, 1; 0, 2, 2, 2, 2, 2, 2, 0.

The proof is divided into five lemmas which are preceded by some general remarks concerning steps and paths for $[y^2]$. In making a step, we use one of the two expansions;

$$(y^2)_{2i} = \sum_{j=0}^{i} c_j y_{i-j} y_{i+j}$$

where, for $j=1, \cdots, i$, $c_j=2C_j^{2i}$ and $c_0=C_0^{2i}$,

$$(y^2)_{2i+1} = \sum_{j=0}^{i} d_j y_{i-j} y_{i+1+j}$$

where, for all j, $d_j=2C_j^{2i+1}$. All coefficients except c_0 are twice a binomial coefficient which means that most second multipliers are minus one, while -2, and $-1/2$ are the second multipliers for steps which respectively start and end at $(y_i)^2$.

Let P be any pp. with excess weight zero or one. Since there is a unique α-term A with the same degree and weight as P, $P \equiv cA$ (mod $[y^2]$), with c a rational number. $m(P)=c/m(P, A)$ will be called the multiplier of P, where $m(P, A)$ is the first multiplier of a path which starts at P and ends at A. Let $\mathfrak{R}$ be the fixed set of paths which occur in any specific reduction of P. From Lemma I′ of the first section, it is clear that $m(P)$ is the sum of the second multipliers of all the paths in $\mathfrak{R}$. Since $P \in [y^2]$ if and only if this sum is zero, it will be sufficient when carrying out a reduction to take note of the second multipliers of the paths which arise in the particular reduction developed.

LEMMA I. *$m(a_1, \cdots, a_n, 0) = m(0, \cdots, 0, a_1, \cdots, a_n, 0)$ [t zeros before a_1].*

PROOF. $y_{i_1}y_{i_2}y_{i_3} \cdots y_{i_{n+1}}$ has the weight sequence $(a_1, \cdots, a_n, 0)$ if $i_j = 2(j-1)+a_j-a_{j-1}$; and $yy_{i_1+2} \cdots y_{i_n+2}$ has the weight sequence $(0, a_1, \cdots, a_n, 0)$. If $(a_1, \cdots, a_n, 0) = -\sum c_i Q_i$, let $\overline{Q}_i$, for each i, be the pp. Q_i with the derivatives of each y increased by two. $(d_1, \cdots, d_n, 0)$ being the weight sequence of Q_i, the weight sequence of $y\overline{Q}_i$ is $(0, d_1, \cdots, d_n, 0)$ and $(0, a_1, \cdots, a_n, 0) \equiv -y(\sum c_i' \overline{Q}_i)$ for suitably chosen constants c_i'. Since $M((a_1, \cdots, a_n, 0), Q_i) = M((0, a_1, \cdots, a_n, 0), \overline{Q}_i)$, $m(a_1, \cdots, a_n, 0) = m(0, a_1, \cdots, a_n, 0)$. An induction proof provides the justification of the statement of the lemma.

COROLLARY. *$m(a_1, \cdots, a_t, 0, \cdots, 0, b_1, \cdots, b_n, 0)$ [k zeros between a_t and b_1] $= m(a_1, \cdots, a_t, 0)m(b_1, \cdots, b_n, 0)$ for $k>0$.*

From the definition of the multiplier of a pp., and there being a unique α-term for each degree of excess weight zero: $(a_1, \cdots, a_t, 0, \cdots, 0, b_1, \cdots, b_n, 0)$ [k zeros between a_t and b_1] $\equiv m(a_1, \cdots, a_t, 0)$ $(0, \cdots, 0, b_1, \cdots, b_n, 0)$ [$t+m$ zeros before b_1] $\equiv m(a_1, \cdots, a_t, 0)$ $m(b_1, \cdots, b_n, 0)$ $(0, \cdots, 0)$ [$t+k+n+1$ zeros]. Therefore, $m(a_1, \cdots, a_t, 0, \cdots, 0, b_1, \cdots, b_n, 0)$ [k zeros between a_t and b_1] $= m(a_1, \cdots, a_t, 0)m(b_1, \cdots, b_n, 0)$.

$A \equiv^M c_i B_i$ where the A and B_i are pp. will be used to mean that $M(A, B_i) = c_i$ and $A \equiv \sum(c_i m(A, B_i)B_i)$.

LEMMA II. (i) *$m(1, \cdots, 1, S, 0)$ [n ones before S] $= (-1)^{n-1} \cdot m(1, S, 0)$ for any sequence S.*

(ii) *$m(S, 1) = (-1/2)m(S, 1, 0)$ for any sequence S.*

Part (i) is true for $n=1$. $P=y_1y_2\cdots$ has a weight sequence $(1,\cdots,1,S,0)$ [n ones before S] and $P\equiv^M -yy_3\cdots$ which has the weight sequence $(0,1,\cdots,1,S,0)$ [$n-1$ ones before S]. Thus $m(1,\cdots,S,0)$ [n ones before S]$=-m(0,1,\cdots,1,S,0)$[$n-1$ ones before S]$=-(-1)^{n-2}m(1,S,0)=(-1)^{n-1}m(1,S,0)$.

For part (ii) let $S=(a_1,\cdots,a_n)$. There is a unique α-term of degree $n+1$ and excess weight one, namely, $yy_2y_4\cdots y_{2n-2}y_{2n+1}$. Thus $(a_1,\cdots,a_n,1)\equiv^M kyy_2\cdots y_{2n-2}y_{2n+1}$. However $(a_1,\cdots,a_n,1)\cdot y_{2n+1}$ has the weight sequence $(a_1,\cdots,a_n,1,0)$; and $(a_1,\cdots,a_n,1,0)\equiv^M kyy_2\cdots y_{2n-2}y_{2n+1}^2\equiv^M k(-2)yy_2\cdots y_{2n}y_{2n+2}$. Thus $m(a_1,\cdots,a_n,1,0)=-2k$ and $k=m(a_1,\cdots,a_n,1)=(-1/2)m(a_1,\cdots,a_n,1,0)$. The demonstration of the lemma is complete.

We also note that $m(a_1,\cdots,a_n,0)=m(a_n,a_{n-1},\cdots,a_1,0)$. The proof consists in noting that the mapping $y_i\to y_{2n+2-i}$ provides a transition from the first pp. to the second, as well as a transition from an M-congruence relation ($\equiv^M$) used in the reduction of the first pp. to a valid M-congruence relation for the second.

It will be necessary to use the following multiples:

(i) $m(1, 0) = 2$, (iv) $m(2, 2, 0) = -6$,

(ii) $m(1, 2, 1, 0) = -4$, (v) $m(1, 2, 2, 1, 0) = 0$.

(iii) $m(1, 2, 2, 0) = 4$,

(i) y_1^2 has the weight sequence $(1,0)$. $y_1^2\equiv^M -2yy_2$ or $m(1,0)=2$.

(ii) $y_1y_3^2y_5$ has the weight sequence $(1,2,1,0)$. $y_1y_5(y_3)^2\equiv^M y_1y_5(-2y_1y_5-2y_2y_4)$ or in terms of sequences, $(1,2,1,0)\equiv^M -2(1,0,1,0)-2(1,1,1,0)$. Hence $m(1,2,1,0)=-2\cdot 4-(2(-2))=-4$.

(iii) $y_1y_3y_4^2$ has the weight sequence $(1,2,2,0)$. $y_1y_3(y_4)^2\equiv^M y_1y_3(-2y_2y_6-2y_3y_5)$, or $(1,2,2,0)\equiv^M -2(1,1,0,0)-2(1,2,1,0)$. Hence $m(1,2,2,0)=-2\cdot 2-2(-4)=4$.

(iv) y_2^3 has the weight sequence $(2,2,0)$. $y_2(y_2)^2\equiv^M y_2(-2yy_4-2y_1y_3)$, or $(2,2,0)\equiv^M -2(0,0,0)-2(1,1,0)$. Hence $m(2,2,0)=-2\cdot 1-2\cdot 2=-6$.

(v) $y_1y_3y_4y_5y_7$ has the weight sequence $(1,2,2,1,0)$. $y_1y_3y_7(y_4y_5)\equiv^M y_1y_3y_7(-y_2y_7-y_3y_6)$, or $(1,2,2,1,0)\equiv^M -(1,1,0,1,0)-(1,2,1,1,0)$. Hence $m(1,2,2,1,0)=-2(-2)-(-(-4))=0$.

Here we encounter a "peculiar" pp., one with a non-negative weight sequence, which is in the ideal. The presence of the pp. in the ideal is sufficient to indicate that for $[y^2]$ the methods of the previous section will not suffice. We continue in order to characterize the "peculiar" pp. which have small positive entries in their weight sequences.

Let $m(1,2_1,2_2,\cdots,2_a,1,0)=f_2(a)$ [where $2_i=2$], $m(2_1,2_2,\cdots,$

$2_a, 1, 0) = f_1(a)$ [where $2_i = 2$] and $m(2_1, 2_2, \cdots, 2_a, 0) = f_0(a)$ [where $2_i = 2$]. Due to the inequalities the entries in a sequence must satisfy, we note that $f_0(a)$ and $f_1(a)$ are defined only for $a \geqq 2$.

LEMMA III.

(i) *For* $a \geqq 3$: $f_2(a) = 2f_1(a-1) + f_2(a-1)$ *and* $f_1(a) = -2f_1(a-1) - 2f_2(a-1)$.

(ii) $f_1(a) = 2f_2(a-2)$ *for* $a \geqq 4$, *and* $f_2(a) = 4f_2(a-3) + f_2(a-1)$ *for* $a \geqq 5$.

(iii) $f_1(a) = 2f_0(a-1) + f_1(a-1)$ *and* $f_0(a) = -2f_0(a-1) - 2f_1(a-1)$ *for* $a \geqq 3$, *and* $f_0(a) = 2f_1(a-2)$ *for* $a \geqq 4$.

PROOF OF PART (i). $(1, 2_1, 2_2, \cdots, 2_a, 1, 0)$ [where $2_i = 2$] is the weight sequence of $y_1y_3y_4y_6 \cdots$. $(y_3y_4)y_1y_6 \cdots \equiv^M y_1y_6 \cdots (-y_1y_6 - y_2y_5)$, or $(1, 2_1, 2_2, \cdots, 2_a, 1, 0) \equiv^M -(1, 0, 2_1, 2_2, \cdots, 2_{a-1}, 1, 0) - (1, 1, 2_1, 2_2, \cdots, 2_{a-1}, 1, 0)$ [where $2_i = 2$]. Thus $m(1, 2_1, 2_2, \cdots, 2_a, 1, 0) = 2m(2_1, 2_2, \cdots, 2_{a-1}, 1, 0) + m(1, 2_1, 2_2, \cdots, 2_{a-1}, 1, 0)$ [where $2_i = 2$] or $f_2(a) = 2f_1(a-1) + f_2(a-1)$. $(2_1, 2_2, \cdots, 2_a, 1, 0)$ [where $2_i = 2$] is the sequence of $y_2^2y_4 \cdots$. $(y_2^2)y_4 \cdots \equiv^M y_4 \cdots (-2yy_4 - 2y_1y_3)$, or $(2_1, 2_2, \cdots, 2_a, 1, 0) \equiv^M -2(0, 2_1, 2_2, \cdots, 2_{a-1}, 1, 0) - 2(1, 2_1, 2_2, \cdots, 2_{a-1}, 1, 0)$ [where $2_i = 2$]. Thus $f_1(a) = -2f_1(a-1) - 2f_2(a-1)$.

PROOF OF (ii). $f_1(a) = -2f_1(a-1) - 2f_2(a-1) = -2(-2f_1(a-2) - 2f_2(a-2)) - 2(2f_1(a-2) + f_2(a-2)) = 2f_2(a-2)$. $f_2(a) = 2f_1(a-1) + f_2(a-1) = 2(2f_2(a-3) + f_2(a-1)) = 4f_2(a-3) + f_2(a-1)$.

PROOF OF (iii). $y_1y_3y_4y_6 \cdots$ has the sequence $(1, 2_1, 2_2, \cdots, 2_a, 0)$ [where $2_i = 2$]. $y_1y_6 \cdots (y_3y_4) \equiv^M y_1y_6 \cdots (-y_1y_6 - y_2y_5)$ or $(1, 2_1, 2_2, \cdots, 2_a, 0) \equiv^M -(1, 0, 2_1, 2_2, \cdots, 2_{a-1}, 0) - (1, 1, 2_1, 2_2, \cdots, 2_{a-1}, 0)$ [where $2_i = 2$]. Thus $m(1, 2_1, 2_2, \cdots, 2_a, 0) = 2m(2_1, 2_2, \cdots, 2_{a-1}, 0) + m(1, 2_1, 2_2, \cdots, 2_{a-1}, 0)$ and $m(1, 2_1, 2_2, \cdots, 2_a, 0) = m(2_1, 2_2, \cdots, 2_a, 1, 0) = f_1(a) = 2f_0(a-1) + f_1(a-1)$ [where $2_i = 2$]. $y_2^2y_4 \cdots$ has the sequence $(2_1, 2_2, \cdots, 2_a, 0)$ [where $2_i = 2$]. $(y_2^2)y_4 \cdots \equiv^M y_4 \cdots (-2yy_4 - 2y_1y_3)$ or $(2_1, 2_2, \cdots, 2_a, 0) \equiv^M -2(0, 2_1, 2_2, \cdots, 2_{a-1}, 0) - 2(1, 2_1, 2_2, \cdots, 2_{a-1}, 0)$ [where $2_i = 2$]. Thus $f_0(a) = -2f_0(a-1) - 2f_1(a-1)$. From the previous results it is easily seen that $f_0(a) = 2f_1(a-2)$. With these facts we find: $f_2(3) = 8$, $f_1(3) = -8$, $f_2(4) = -8$ and $f_0(3) = 4$.

LEMMA IV. $f_2(a) = 0$ *if and only if* $a = 2$, $f_1(a) = 0$ *if and only if* $a = 4$, *and* $f_0(a) = 0$ *if and only if* $a = 6$.

PROOF. For $a \geqq 5$, $f_2(a) = 2^3 \cdot$odd number, and thus does not vanish. It is known that $f_2(5) = 4f_2(2) + f_2(4) = 4 \cdot 0 + (-8) = 2^3(-1)$. Then by induction: $f_2(a) = 4f_2(a-3) + f_2(a-1) = 4 \cdot 2^3 \cdot \text{odd} + 2^3 \cdot \text{odd}$

$=2^3\cdot$odd number. With the computation of $f_2(1)$, $f_2(2)$, $f_2(3)$ and $f_2(4)$, we are able to conclude that $f_2(a)$ vanishes if and only if $a=2$.

Noting that $f_1(2)\neq 0$, and $f_1(a)=2f_2(a-2)$, we conclude that $f_1(a)=0$ if and only if $a=4$. In the same manner, $f_0(2)\neq 0$, $f_0(3)\neq 0$ and $f_0(a)=2f_1(a-2)$ imply that $f_0(a)=0$ if and only if $a=6$.

LEMMA V.

(i) $m(1, 2_1, 2_2, \cdots, 2_{a_1}, 1, 2_1, 2_2, \cdots, 2_{a_2}, 1, \cdots, 1, 2_1, 2_2, \cdots, 2_{a_n}, 1, 0)$ [*where* $2_i=2$] $=(-1/2)^{n-1}\prod_{i=1}^n f_2(a_i)$.

(ii) $m(2_1, 2_2, \cdots, 2_{a_1}, 1, 2_1, 2_2, \cdots, 2_{a_2}, 1, \cdots, 1, 2_1, 2_2, \cdots, 2_{a_n}, 1, 0)$ [*where* $2_i=2$] $=(-1/2)^{n-1}f_1(a_1)\prod_{i=2}^n f_2(a_i)$.

(iii) $m(2_1, 2_2, \cdots, 2_{a_1}, 1, 2_1, 2_2, \cdots, 2_{a_2}, 1, \cdots, 1, 2_1, 2_2, \cdots, 2_{a_n}, 0)$ [*where* $2_i=2$] $=(-1/2)^{n-1}f_1(a_1)f_1(a_n)\cdot\prod_{i=2}^{n-1} f_2(a_i)$ *where* $n>1$.

PROOF. We use the fact that for each degree there is only one α-term of excess weight equal to one, in conjunction with (ii) of the second lemma.

Part (i) is true if $n=1$. For larger values: $m(1, 2_1, 2_2, \cdots, 2_{a_1}, 1, 2_1, 2_2, \cdots, 2_{a_2}, 1, \cdots, 1, 2_1, 2_2, \cdots, 2_{a_n}, 1, 0)=(-1/2)\, m(1, 2_1, 2_2, \cdots, 2_{a_1}, 1, 0)$ [where $2_i=2$], $m(1, 2_1, 2_2, \cdots, 2_{a_2}, 1, \cdots, 1, 2_1, 2_2, \cdots, 2_{a_n}, 1, 0)$ [where $2_i=2$] $=(-1/2)f_2(a_1)(-1/2)^{n-2}\cdot\prod_{i=2}^n f_2(a_i)=(-1/2)^{n-1}\prod_{i=1}^n f_2(a_i)$.

The proofs of (ii) and (iii) are carried out in a similar fashion. Reference to the corollary of Lemma I, part (i) of Lemma II and Lemmas IV and V provide the demonstration of the theorem stated at the beginning of the section.

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