MOMENTS OF ANALYTIC FUNCTIONS

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There are many theorems which state that an analytic function which is of sufficiently slow growth in a half-plane and tends sufficiently rapidly to zero along an interior line must vanish identically. Recently a theorem of this sort was proved by San Juan [4] and Sunyer Balaguer [5], where the condition of rapid approach to zero is expressed indirectly by the smallness of a set of moments. In the more precise formulation of Sunyer Balaguer, the theorem is as follows.

THEOREM 1. *If $f(z)$ is regular and bounded for $x \geqq 0$, and*

$$\int_0^\infty |f(x)| x^n dx < \Gamma(\beta n + 1), \qquad \beta < 1,$$

for an infinity of n, then $f(z) \equiv 0$.

This theorem, in a still more general form, can be deduced from a theorem of Ahlfors and Heins [1; 3] on subharmonic functions. Stated in the form appropriate for analytic functions of exponential type, this reads as follows.

THEOREM 2. *If $f(z)$ is regular and of exponential type for $x \geqq 0$, bounded on the imaginary axis, and not identically zero, then for some number c we have $\lim_{r\to\infty} r^{-1} \log |f(re^{i\theta})| = c \cos \theta$, for all θ in $(-\pi/2, \pi/2)$ except a set of outer capacity 0, and for each θ in this interval if r is excluded from a set of finite logarithmic length.*

A function $f(z)$ is of exponential type if $|f(z)| \leqq Ae^{k|z|}$ for some k and A; the logarithmic length of E is $\int_E x^{-1} dx$.

I use the second part of Theorem 2 to prove the following stronger form of Theorem 1.

THEOREM 3. *If $f(z)$ is regular and of exponential type for $x \geqq 0$, and is bounded on the imaginary axis, and if*

$$(1) \qquad \int_0^\infty |f(re^{i\theta})| r^n dr < n^n e^{-n\phi(n)}, \qquad \phi(n) \to \infty,$$

for some θ, $-\pi/2 < \theta < \pi/2$, and for an infinity of n, then $f(z) \equiv 0$.

Theorem 1 is effectively the case in which $\phi(n) = (1-\beta) \log n$.

Presented to the Society, September 3, 1954; received by the editors May 28, 1954 and, in revised form, August 30, 1954.

We may suppose, without loss of generality, that $\phi(n) < \log n$. Let $\alpha(n)$ be a function such that $0 < \alpha(n) \uparrow 1$, and $\{1-\alpha(n)\} \log n \to \infty$, but is $o\{\phi(n)\}$. Let $\mu(E)$ denote the logarithmic length of E, and suppose that $|f(re^{i\theta})| > e^{-n}$ on a set E_n in $(n^{\alpha(n)}, \lambda n^{\alpha(n)})$, $\lambda > 1$, where n is an integer for which (1) holds. We have

$$n^n e^{-n\phi(n)} > \int_{E_n} |f(re^{i\theta})| r^n dr > e^{-n} \int_{E_n} r^{n+1} r^{-1} dr$$
$$> e^{-n} n^{(n+1)\alpha(n)} \mu(E_n)$$

and hence

$$\mu(E_n) < \exp\{n + n \log n - (n+1) \log n + (n+1) o[\phi(n)] - n\phi(n)\}$$
$$= o(1).$$

Thus $|f(re^{i\theta})| \leqq e^{-n}$ on $(n^{\alpha(n)}, \lambda n^{\alpha(n)})$ except at most for a set whose logarithmic length approaches zero as $n \to \infty$ through the values satisfying (1). In other words, in the specified intervals, except for a set of infinitesimal logarithmic length,

$$r^{-1} \log |f(re^{i\theta})| \leqq -n/r \leqq -n/n^{\alpha(n)} = -\exp\{[1-\alpha(n)]\log n\}$$
$$\to -\infty.$$

Since the logarithmic length of $(n^{\alpha(n)}, \lambda n^{\alpha(n)})$ is $\log \lambda$, we have $r^{-1} \log |f(re^{i\theta})| \to -\infty$ on a set of intervals of infinite logarithmic length. By Theorem 2, this can happen only if $f(z) \equiv 0$.

It is not essential to suppose that $f(iy)$ is bounded, since Theorem 2 remains true if we assume only, for example, that

$$\int_{-\infty}^{\infty} (1+y^2)^{-1} \log^+ |f(iy)| dy < \infty$$

(see [2]); for the application to Theorem 3 we need still less, for example that $\int_{-\infty}^{\infty} y^{-2} \log |f(iy)f(-iy)| dy < O(1)$.

References

1. L. Ahlfors and M. Heins, *Questions of regularity connected with the Phragmén-Lindelöf principle*, Ann. of Math. (2) vol. 50 (1949) pp. 341–346.

2. R. P. Boas, Jr., *Asymptotic properties of functions of exponential type*, Duke Math. J. vol. 20 (1953) pp. 433–448.

3. J. Lelong-Ferrand, *Étude au voisinage de la frontière des fonctions surharmoniques positives dans un demi-espace*, Ann. École Norm. (3) vol. 66 (1949) pp. 125–159.

4. R. San Juan, *L'accroissement des moments d'une fonction holomorphe dans un angle*, C. R. Acad. Sci. Paris vol. 236 (1953) pp. 1941–1943.

5. F. Sunyer Balaguer, *Sobre los momentos de las funciones holomorfas y acotadas en un angulo*, Revista Matemática Hispano-Americana (4) vol. 13 (1953) pp. 241–246.

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