MOMENTS OF ANALYTIC FUNCTIONS
R. P. BOAS, JR.

There are many theorems which state that an analytic function which is of sufficiently slow growth in a half-plane and tends sufficiently rapidly to zero along an interior line must vanish identically. Recently a theorem of this sort was proved by San Juan [4] and Sunyer Balaguer [5], where the condition of rapid approach to zero is expressed indirectly by the smallness of a set of moments. In the more precise formulation of Sunyer Balaguer, the theorem is as follows.

Theorem 1. If \( f(z) \) is regular and bounded for \( x \geq 0 \), and

\[
\int_0^\infty |f(x)| x^n dx < \Gamma(\beta n + 1), \quad \beta < 1,
\]

for an infinity of \( n \), then \( f(z) \equiv 0 \).

This theorem, in a still more general form, can be deduced from a theorem of Ahlfors and Heins [1; 3] on subharmonic functions. Stated in the form appropriate for analytic functions of exponential type, this reads as follows.

Theorem 2. If \( f(z) \) is regular and of exponential type for \( x \geq 0 \), bounded on the imaginary axis, and not identically zero, then for some number \( c \) we have \( \lim_{r \to \infty} r^{-1} \log |f(re^{i\theta})| = c \cos \theta \), for all \( \theta \) in \((-\pi/2, \pi/2)\) except a set of outer capacity 0, and for each \( \theta \) in this interval if \( r \) is excluded from a set of finite logarithmic length.

A function \( f(z) \) is of exponential type if \( |f(z)| \leq Ae^{k|z|} \) for some \( k \) and \( A \); the logarithmic length of \( E \) is \( \int_E x^{-1} dx \).

I use the second part of Theorem 2 to prove the following stronger form of Theorem 1.

Theorem 3. If \( f(z) \) is regular and of exponential type for \( x \geq 0 \), and is bounded on the imaginary axis, and if

\[
(1) \quad \int_0^\infty |f(re^{i\theta})| r^n dr < n^n e^{-n\phi(n)}, \quad \phi(n) \to \infty,
\]

for some \( \theta \), \(-\pi/2 < \theta < \pi/2\), and for an infinity of \( n \), then \( f(z) \equiv 0 \).

Theorem 1 is effectively the case in which \( \phi(n) = (1 - \beta) \log n \).

Presented to the Society, September 3, 1954; received by the editors May 28, 1954 and, in revised form, August 30, 1954.

412
We may suppose, without loss of generality, that \( \phi(n) < \log n \). Let \( \alpha(n) \) be a function such that \( 0 < \alpha(n) \uparrow 1 \), and \( \{1 - \alpha(n)\} \log n \to \infty \), but is \( o(\phi(n)) \). Let \( \mu(E) \) denote the logarithmic length of \( E \), and suppose that \( |f(re^{i\theta})| > e^{-n} \) on a set \( E_n \) in \( (n\alpha(n), \lambda n\alpha(n)) \), \( \lambda > 1 \), where \( n \) is an integer for which (1) holds. We have

\[
n^n e^{-n\phi(n)} > \int_{E_n} |f(re^{i\theta})| r^n dr > e^{-n} \int_{E_n} r^{n+1} r^{-1} dr
\]

and hence

\[
\mu(E_n) < \exp \{ n + n \log n - (n + 1) \log n + (n + 1)\phi(n) - n\phi(n) \}
\]

\( = o(1) \).

Thus \( |f(re^{i\theta})| \leq e^{-n} \) on \( (n\alpha(n), \lambda n\alpha(n)) \) except at most for a set whose logarithmic length approaches zero as \( n \to \infty \) through the values satisfying (1). In other words, in the specified intervals, except for a set of infinitesimal logarithmic length,

\[
r^{-1} \log |f(re^{i\theta})| \leq -n/r \leq -n/n\alpha(n) = -\exp \{ [1 - \alpha(n)]\log n \}
\]

\( \to -\infty \).

Since the logarithmic length of \( (n\alpha(n), \lambda n\alpha(n)) \) is \( \log \lambda \), we have \( r^{-1} \log |f(re^{i\theta})| \to -\infty \) on a set of intervals of infinite logarithmic length. By Theorem 2, this can happen only if \( f(z) \equiv 0 \).

It is not essential to suppose that \( f(iy) \) is bounded, since Theorem 2 remains true if we assume only, for example, that

\[
\int_{-\infty}^{\infty} (1 + y^2)^{-1} \log^+ |f(iy)| dy < \infty
\]

(see [2]); for the application to Theorem 3 we need still less, for example that \( \int_{-\infty}^{\infty} y^{-2} \log |f(iy)f(-iy)| dy < O(1) \).

**References**


