A NOTE ON TAUBERIAN CONDITIONS FOR ABEL AND CESÀRO SUMMABILITY

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A conjecture has recently been made by P. L. Butzer\footnote{Bull. Amer. Math. Soc. Research Problem 60-3-14.} that there exist Tauberian conditions for the Cesàro method of summability which are not also Tauberian conditions for Abel summability.

It was pointed out\footnote{In a letter to the editors received July 14, 1954.} by G. Lorentz that the conjecture is true in a trivial sense if we define the Tauberian condition as membership of a class $T$ consisting of all series satisfying one of the classical Tauberian conditions for Cesàro summation ($na_n \geq -K$, for example) together with one particular series which is Abel, but not Cesàro, summable. It does not seem possible to restrict the class of admissible Tauberian conditions in a natural way so that examples of this rather artificial kind are excluded.

However, even if Butzer's question does not admit a nontrivial yes or no answer, it still has substance if we look for a solution with some intrinsic formal interest.

The condition $(\tau)$ defined below applies to the slightly more general summation methods defined by integral transforms of functions of a continuous real variable. These reduce to the classical methods of summation of series when the functions are step functions with jumps at integral values. Thus, if $s(x)$ is defined for $x \geq 0$ and is integrable over any finite interval, it tends to $A$ in the Abel sense if

\begin{equation}
\lim_{\delta \to 0} \delta \int_{0}^{\infty} e^{-\delta s(x)} dx = A,
\end{equation}

it being assumed that the integral converges absolutely when $\delta > 0$. It tends to $A$ in the Cesàro sense if

\begin{equation}
\lim_{x \to \infty} \frac{1}{x} \int_{0}^{x} s(x) dx = A.
\end{equation}

A Tauberian condition for Abel or Cesàro summability is one which, together with (1) or (2), respectively, ensures that $s(x) \to A$. We shall say that $s(x)$ satisfies the condition $(\tau)$ if, for every

Received by the editors September 30, 1954 and, in revised form, November 8, 1954.

2 In a letter to the editors received July 14, 1954.

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\( \epsilon > 0 \), there is a positive number \( \eta(\epsilon) \), independent of \( x \), such that, for all sufficiently large \( x \),

\[
\left| \int_x^{xR} [s(y) - s(x)] dy \right| \leq (R - 1)\epsilon x
\]

for some \( X(\epsilon, x) \), \( R = R(\epsilon, x) \) satisfying

\[
R \geq 1 + \eta, \quad xR^{-1} \leq X \leq x.
\]

The essential point about condition (\( \tau \)) is that it restricts the mean values of \( s(y) - s(x) \) and not \( |s(y) - s(x)| \), as do the familiar classical conditions. It is easy to show that the stronger form of (\( \tau \)) with the modulus inside the integral is a Tauberian condition for Abel summation and therefore not an answer to Butzer’s problem.

In the case of a step function \( s(x) = \sum_{n \leq x} a_n \), it is easy to show that (\( \tau \)) is implied by the classical Tauberian condition \( na_n \geq -K \) for some positive constant \( K \) or by the “high indices” condition.

**Theorem 1.** Condition (\( \tau \)) is a Tauberian condition for Cesàro summability.

Writing

\[
C(X) = \frac{1}{X} \int_0^X s(x) dx,
\]

it follows from (3) that

\[
\left| s(x) - \frac{R}{R - 1} C(RX) + \frac{C(X)}{R - 1} \right| < \epsilon
\]

for large \( x \) and assuming, by a trivial transformation, that \( A = 0 \), \( C(x) \to 0 \), it follows that

\[
\limsup |s(x)| < \epsilon, \quad s(x) \to 0.
\]

To conclude, we show that (\( \tau \)) is not an Abel-Tauberian condition by proving:

**Theorem 2.** There exists a function \( s(x) \) satisfying (\( \tau \)) which does not tend to a limit as \( x \to \infty \) but which tends to a limit in the Abel sense.

We write

\[
\lambda_m = (2m + 1) \log (2m + 1) \quad (m = 0, 1, 2, \ldots)
\]

and define
(7) \[ s(x) = (-2)^m \quad \text{for} \quad \lambda_m \leq x < \lambda_{m+1}. \]

It is plain that \( s(x) \) does not tend to a limit, but

\[
\delta \int_0^\infty e^{-\delta x} s(x) \, dx = \int_0^\infty e^{-\delta x} s(x) \, dx = -3 \sum_{m=0}^\infty e^{-\delta (2m+1) \log (2m+1)(-2)^{m-1}}
\]

\[
= -3 \sum_{n=1}^\infty e^{-\delta n} \log 2 \frac{(n-2)^{1/2}}{n} \cos (n+1)\pi/2
\]

\[
= R \left[ -3 \cdot 2^{-3/2} e^{\pi/2} \sum_{n=1}^\infty \frac{e^{-\delta n} \log (2^{1/2} e^{\pi/2})^n}{n} \right],
\]

and it is known\(^8\) that this tends to a limit as \( \delta \to +0 \).

Finally, we have to show that \( s(x) \) satisfies (\( \tau \)). For every positive \( \epsilon \), we define

\[
X(\epsilon, x) = \lambda_M,
\]

where \( M = M(x) \) is determined by

(9) \[ \lambda_M \leq x < \lambda_{M+1}. \]

We shall suppose that \( M \) is even. The odd case needs only trivial changes. Next we define the even integer \( P = P(x) \) by

(10) \[ \lambda_{P-1} \leq 2\lambda_M < \lambda_{P+1}. \]

For all positive \( \epsilon \), we shall define \( R(\epsilon, x) = R(x) \) so that

\[
(11) \quad \lambda_P \leq R\lambda_M < \lambda_{P+1}.
\]

Since \( \lambda_{m+1} - \lambda_m = o(\lambda_m) \), it follows from (10) and (11) that \( R(x) \to 2 \) as \( x \to \infty \); and therefore (4) holds. Now

\[
\int_{\lambda_M}^{\lambda_P} s(y) \, dy = \sum_{m=M}^{P-1} (\lambda_{m+1} - \lambda_m)(-2)^m
\]

\[
< (-2)^{P-1}(\lambda_P - \lambda_{P-1}) \left( 1 - \sum_{v=1}^{P-M-1} 2^{-v} \right)
\]

since \( P \) is even and \( \lambda_{m+1} - \lambda_m \) increases with \( m \). This expression is negative, and therefore

\[
(12) \quad \int_{\lambda_M}^{\lambda_P} s(y) \, dy < 0 < s(x).
\]

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A similar argument shows that
\[ \int_{\lambda_M}^{\lambda_{P-1}} s(y) dy > 0, \]
so that
\[ \int_{\lambda_M}^{\lambda_{P-1}} s(y) dy > \int_{\lambda_{P-1}}^{\lambda_{P-1}} s(y) dy = 2^P (\lambda_{P+1} - \lambda_P) - 2^{P-1} (\lambda_P - \lambda_{P-1}) \]
\[ > 2^{P-1} (\lambda_{P+1} - \lambda_P) > \frac{2^{P-M-1}}{P - M + 1} (\lambda_{P+1} - \lambda_M) s(x) \]
since \( \lambda_{m+1} - \lambda_m \) increases and \( s(x) = 2^M \). Moreover, since \( \lambda_{m+1} - \lambda_m = o(\lambda_m) \), it follows from (10) that \( P - M \to \infty \) as \( x \to \infty \) and so \( 2^{P-M-1} > P - M + 1 \) for large \( x \). Hence, writing \( X \) for \( \lambda_M \),

\[ (13) \quad \frac{1}{(R - 1)X} \int_X^{RX} s(y) dy > s(x) \]
when \( XR = \lambda_{P+1} \). Since the expression on the left of (13) is continuous in \( R \) for \( \lambda_P \leq XR \leq \lambda_{P+1} \), and is less than \( s(x) \) when \( XR = \lambda_P \), by (12), it follows that

\[ (14) \quad \frac{1}{(R - 1)X} \int_X^{RX} s(y) dy = s(x) \]
for some \( R \) in the range defined by (11). If we define \( R(x) \) to be this value, the condition \((\tau)\) is satisfied.

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