A NOTE ON TAUBERIAN CONDITIONS FOR ABEL AND CESÀRO SUMMABILITY

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A conjecture has recently been made by P. L. Butzer[1] that there exist Tauberian conditions for the Cesàro method of summability which are not also Tauberian conditions for Abel summability.

It was pointed out[2] by G. Lorentz that the conjecture is true in a trivial sense if we define the Tauberian condition as membership of a class T consisting of all series satisfying one of the classical Tauberian conditions for Cesàro summation ($na_n \geqq -K$, for example) together with one particular series which is Abel, but not Cesàro, summable. It does not seem possible to restrict the class of admissible Tauberian conditions in a natural way so that examples of this rather artificial kind are excluded.

However, even if Butzer's question does not admit a nontrivial yes or no answer, it still has substance if we look for a solution with some intrinsic formal interest.

The condition (τ) defined below applies to the slightly more general summation methods defined by integral transforms of functions of a continuous real variable. These reduce to the classical methods of summation of series when the functions are step functions with jumps at integral values. Thus, if $s(x)$ is defined for $x \geqq 0$ and is integrable over any finite interval, it tends to A in the Abel sense if

$$\lim_{\delta \to +0} \delta \int_0^\infty e^{-\delta x} s(x) dx = A, \tag{1}$$

it being assumed that the integral converges absolutely when $\delta > 0$. It tends to A in the Cesàro sense if

$$\lim_{X \to \infty} \frac{1}{X} \int_0^X s(x) dx = A. \tag{2}$$

A Tauberian condition for Abel or Cesàro summability is one which, together with (1) or (2), respectively, ensures that $s(x) \to A$.

We shall say that $s(x)$ satisfies the condition (τ) if, for every

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[1] Bull. Amer. Math. Soc. Research Problem 60-3-14.

[2] In a letter to the editors received July 14, 1954.

$\epsilon > 0$, there is a positive number $\eta(\epsilon)$, independent of x, such that, for all sufficiently large x,

$$\left| \int_X^{RX} [s(y) - s(x)] dy \right| \leqq (R - 1)\epsilon X \tag{3}$$

for some $X(\epsilon, x)$, $R = R(\epsilon, x)$ satisfying

$$R \geqq 1 + \eta, \tag{4}$$

$$xR^{-1} \leqq X \leqq x. \tag{5}$$

The essential point about condition (τ) is that it restricts the mean values of $s(y) - s(x)$ and not $|s(y) - s(x)|$, as do the familiar classical conditions. It is easy to show that the stronger form of (τ) with the modulus inside the integral is a Tauberian condition for Abel summation and therefore not an answer to Butzer's problem.

In the case of a step function $s(x) = \sum_{n \leqq x} a_n$, it is easy to show that (τ) is implied by the classical Tauberian condition $na_n \geqq -K$ for some positive constant K or by the "high indices" condition.

THEOREM 1. *Condition (τ) is a Tauberian condition for Cesàro summability.*

Writing

$$C(X) = \frac{1}{X} \int_0^X s(x) dx,$$

it follows from (3) that

$$\left| s(x) - \frac{R}{R - 1} C(RX) + \frac{C(X)}{R - 1} \right| < \epsilon$$

for large x and assuming, by a trivial transformation, that $A = 0$, $C(x) \to 0$, it follows that

$$\limsup |s(x)| < \epsilon, \qquad s(x) \to 0.$$

To conclude, we show that (τ) is not an Abel-Tauberian condition by proving:

THEOREM 2. *There exists a function $s(x)$ satisfying (τ) which does not tend to a limit as $x \to \infty$ but which tends to a limit in the Abel sense.*

We write

$$\lambda_m = (2m + 1) \log (2m + 1) \qquad (m = 0, 1, 2, \cdots) \tag{6}$$

and define

$$s(x) = (-2)^m \quad \text{for} \quad \lambda_m \leqq x < \lambda_{m+1}. \tag{7}$$

It is plain that $s(x)$ does not tend to a limit, but

$$\delta \int_0^\infty e^{-\delta x} s(x)dx = \int_0^\infty e^{-\delta x} ds(x) = -3\sum_{m=0}^\infty e^{-\delta(2m+1)\log(2m+1)}(-2)^{m-1}$$

$$= -3\sum_{n=1}^\infty e^{-\delta n \log n} 2^{(n-3)/2} \cos(n+1)\pi/2$$

$$= R\left[-3\cdot 2^{-3/2}e^{\pi i/2}\sum_{n=1}^\infty e^{-\delta n\log n}(2^{1/2}e^{\pi i/2})^n\right],$$

and it is known[3] that this tends to a limit as $\delta \to +0$.

Finally, we have to show that $s(x)$ satisfies (τ). For every positive ϵ, we define

$$X(\epsilon, x) = \lambda_M, \tag{8}$$

where $M = M(x)$ is determined by

$$\lambda_M \leqq x < \lambda_{M+1}. \tag{9}$$

We shall suppose that M is even. The odd case needs only trivial changes. Next we define the even integer $P = P(x)$ by

$$\lambda_{P-1} \leqq 2\lambda_M < \lambda_{P+1}. \tag{10}$$

For all positive ϵ, we shall define $R(\epsilon, x) = R(x)$ so that

$$\lambda_P \leqq R\lambda_M < \lambda_{P+1}. \tag{11}$$

Since $\lambda_{m+1} - \lambda_m = o(\lambda_m)$, it follows from (10) and (11) that $R(x) \to 2$ as $x \to \infty$; and therefore (4) holds. Now

$$\int_{\lambda_M}^{\lambda_P} s(y)dy = \sum_{m=M}^{P-1} (\lambda_{m+1} - \lambda_m)(-2)^m$$

$$< (-2)^{P-1}(\lambda_P - \lambda_{P-1})\left(1 - \sum_{v=1}^{P-M-1} 2^{-v}\right)$$

since P is even and $\lambda_{m+1} - \lambda_m$ increases with m. This expression is negative, and therefore

$$\int_{\lambda_M}^{\lambda_P} s(y)dy < 0 < s(x). \tag{12}$$

[3] Lindelöf, Journal de Mathématiques (5) vol. 9 (1903) pp. 213–221. See also Hardy, *Divergent series*, Oxford, 1949, Theorem 32, p. 78.

A similar argument shows that

$$\int_{\lambda_M}^{\lambda_{P-1}} s(y)dy > 0,$$

so that

$$\int_{\lambda_M}^{\lambda_{P-1}} s(y)dy > \int_{\lambda_{P-1}}^{\lambda_{P-1}} s(y)dy = 2^P(\lambda_{P+1} - \lambda_P) - 2^{P-1}(\lambda_P - \lambda_{P-1})$$

$$> 2^{P-1}(\lambda_{P+1} - \lambda_P) > \frac{2^{P-M-1}}{P - M + 1}(\lambda_{P+1} - \lambda_M)s(x)$$

since $\lambda_{m+1} - \lambda_m$ increases and $s(x) = 2^M$. Moreover, since $\lambda_{m+1} - \lambda_m = o(\lambda_m)$, it follows from (10) that $P - M \to \infty$ as $x \to \infty$ and so $2^{P-M-1} > P - M + 1$ for large x. Hence, writing X for λ_M,

$$\frac{1}{(R-1)X}\int_X^{RX} s(y)dy > s(x) \tag{13}$$

when $XR = \lambda_{P+1}$. Since the expression on the left of (13) is continuous in R for $\lambda_P \leqq XR \leqq \lambda_{P+1}$, and is less than $s(x)$ when $XR = \lambda_P$, by (12), it follows that

$$\frac{1}{(R-1)X}\int_X^{RX} s(y)dy = s(x) \tag{14}$$

for some R in the range defined by (11). If we define $R(x)$ to be this value, the condition (τ) is satisfied.

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