PERMUTATIONS PRESERVING CONVERGENCE OF SERIES

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1. **Introduction.** We shall solve a problem involving rearrangements of series which was proposed to the author by R. Creighton Buck. The problem is to characterize those permutations $p_1, p_2, p_3, \cdots$ of the integers $1, 2, 3, \cdots$ such that the two series in

$$\sum_{k=1}^{\infty} a_{p_k} = \sum_{k=1}^{\infty} a_k$$

converge to equal values whenever $\sum a_k$ is a convergent series of real (or complex) terms. We shall show that a given permutation $p_1, p_2, \cdots$ has the required property if and only if there is an integer N such that for each $n = 1, 2, 3, \cdots$ the set of integers which appear among the integers $p_1, p_2, \cdots, p_n$ is representable as the union of N or fewer blocks of consecutive integers.

2. **Proof.** Let $p_1, p_2, \cdots$ be a given permutation of the integers $1, 2, 3, \cdots$. Let

$$s(n) = \sum_{k=1}^{n} a_k, \qquad \sigma(n) = \sum_{k=1}^{n} a_{p_k}.$$

Suppose that n is so great that the integer 1 is included in the set $p_1, p_2, \cdots, p_n$. Then the integers in the set $p_1, p_2, \cdots, p_n$ are, in increasing order,

$$1, 2, 3, \cdots, \beta_0^{(n)}, \alpha_1^{(n)} + 1, \alpha_1^{(n)} + 2, \cdots, \beta_1^{(n)}, \alpha_2^{(n)} + 1,$$
$$\alpha_2^{(n)} + 2, \cdots, \beta_2^{(n)}, \cdots, \alpha_{j_n}^{(n)} + 1, \alpha_{j_n}^{(n)} + 2, \cdots, \beta_{j_n}^{(n)}$$

where

$$0 < \beta_0^{(n)} < \alpha_1^{(n)} < \beta_1^{(n)} < \alpha_2^{(n)} < \beta_2^{(n)} < \cdots < \beta_{j_n}^{(n)}.$$

Hence

$$\sigma(n) = s(\beta_0^{(n)}) + [s(\beta_1^{(n)}) - s(\alpha_1^{(n)})] + \cdots + [s(\beta_{j_n}^{(n)}) - s(\alpha_{j_n}^{(n)})]$$

and therefore

$$\sigma(n) = s(\beta_0^{(n)}) - s(\alpha_1^{(n)}) + s(\beta_1^{(n)}) - \cdots - s(\alpha_{j_n}^{(n)}) + s(\beta_{j_n}^{(n)}).$$

Since $\lim_{n\to\infty} \beta_0^{(n)} = \infty$, this has the standard form

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$$\sigma(n) = \sum_{k=1}^{\infty} a_{nk} s(k)$$

where

$$\lim_{n \to \infty} a_{nk} = 0, \qquad k = 1, 2, 3, \cdots,$$

and

$$\sum_{k=1}^{\infty} a_{nk} = 1, \qquad n = 1, 2, 3, \cdots.$$

This transformation from $s(n)$ to $\sigma(n)$ is regular, that is, such that existence of $\lim s(n)$ implies $\lim \sigma(n) = \lim s_n$, if and only if there is a constant M for which

$$\sum_{k=1}^{\infty} | a_{nk} | \leqq M.$$

We see that

$$\sum_{k=1}^{\infty} | a_{nk} | = 2B_n - 1$$

where B_n is the number of disjoint blocks of consecutive integers appearing in the set $p_1, p_2, \cdots, p_n$ and hence that existence of $\lim s(n)$ implies $\lim \sigma(n) = \lim s(n)$ if and only if there is a constant M for which $2B_n - 1 \leqq M$ or $B_n \leqq (M+1)/2$. This proves the result.

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