COMPACT TRANSFORMATIONS AND THE \( k \)-TOPOLOGY 
IN HILBERT SPACE\(^1\)

RALPH A. RAIMI

1. Introduction. We shall be concerned only with Hilbert spaces, though many of the notions involved can be extended to—and indeed sometimes originally appeared in—a wider context. By a Hilbert space \( H \) will be meant an inner product space of arbitrary dimension, which is moreover complete. The \( k \)-topology for \( H \), originally defined in \([1]\),\(^2\) is generated from a basis set of neighborhoods of the identity \( \theta \), \( \{ V(\theta) \} \), obtained in the following way: If \( K \) is any compact subset of \( H \), let \( V(\theta) = \{ x \in H \mid |\langle x, y \rangle| \leq 1 \text{ for all } y \in K \} \). The interest of the \( k \)-topology lies in the fact that it is the strongest locally convex topology for \( H \) which coincides with the weak topology on all spheres \([2]\).

The purpose of this paper is to introduce an equivalent method of defining this topology, the equivalence to be proved via a lemma concerning compact transformations on \( H \) to \( H \).

2. Compact transformations and sets in \( H \). Lemma 1 is probably well known, being an explicit form of some more general theorems concerning pointwise and uniform convergence of sequences of continuous mappings of a compact set.

**Lemma 1.** Let \( H \) be a Hilbert space with a denumerable orthonormal base \( (e_1, e_2, \cdots, e_n, \cdots) \) in terms of which every element \( x \in H \) has the unique expansion \( x = \sum_{n=1}^{\infty} a_n(x) e_n \). Let \( K \) be a closed bounded subset of \( H \). Then \( K \) is compact if and only if the following criterion holds: For every real \( \epsilon > 0 \), there exists an integer \( N(\epsilon) \) such that \( \| \sum_{i=1}^{\infty} a_i(x) e_i \| < \epsilon \) for all \( x \) in \( K \).

A linear transformation \( T: H \to H \) is called compact if \( \text{Cl}(T(S)) \), the closure of \( T(S) \), is compact, where \( S \) is the unit ball \( \{ x \in H \mid \| x \| \leq 1 \} \).

**Lemma 2.** Let \( H \) be any Hilbert space, and \( K \) any compact subset. Then there exists a compact linear transformation \( T \) such that \( T(S) \supseteq K \).

---

\(^1\) This paper is a short extract from a thesis presented to the University of Michigan for the degree of Ph.D. The work was supported in part by the Office of Naval Research, Contract Nonr-330(00).

\(^2\) Numbers in brackets refer to the bibliography at the end of the paper.

Presented to the Society, October 30, 1954; received by the editors October 6, 1954.
Proof. We may assume without loss of generality that (1) K is closed, convex, symmetric, and (2) that $K \subseteq S$. By symmetric is meant that if $x \in K$, and if $a$ is a scalar such that $|a| = 1$, then $ax \in K$. (1) is possible because if $K$ is extended to its least convex, symmetric, closed hull, the result is still compact. Proving the lemma for this hull proves it a fortiori for $K$. (2) follows from the fact that a scalar multiple of a compact transformation is again compact.

Let $a_1 = \sup \{ \|x\| | x \in K \}$. Since $K$ is compact, and $\|x\|$ is a continuous function on $K$, there is a vector $y_1 \in K$ at which the norm $a_1$ is taken on. Set $e_1 = y_1 / a_1$. Then $e_1$ is the first element of an orthonormal sequence constructed inductively as follows: Denote by $V_n$ the linear extension of $\{ e_1 \cup e_2 \cup \cdots \cup e_n \}$, and by $V_n^\perp$ the orthogonal complement of $V_n$ in $H$. For any element $x \in K$, there is an unique decomposition $x = x^n + u^n$, with $x^n \in V_n$ and $u^n \in V_n^\perp$. Then set $a_{n+1} = \sup \{ \|u^n\| | x \in K, x = x^n + u^n \}$. Since $u^n$ and hence $\|u^n\|$ is a continuous function on the compact $K$, there exists a vector $x = x^n + u^n$ at which this supremum is taken on. Let $y_{n+1} = u^n$ for this $x$ and set $e_{n+1} = y_{n+1} / a_{n+1}$, where $a_{n+1}$ is the supremum in question. Clearly the vectors $y_n$, and hence the unit vectors $e_n$, form an orthogonal set, with $y_{n+1}$ and hence $e_{n+1}$ orthogonal to $V_n$. It is also clear from the construction that $a_{n+1} \leq a_n$ for all $n$. Moreover, $\lim_n a_n = 0$, for if this were not the case, there is some $\delta > 0$ such that $a_n > \delta$ for all $n$. But this is to say that in the construction, if $x_n$ is the sequence in $K$ at which the successive maxima are taken on, and if we represent $x_n = x^n + u^n$ as above, then $\|u^n\| > \delta$ for all $n$. Then for $n > m$, $\|x_m - x_n\| = \|x^n - x^m + u^m - u^n\| \geq \|u^n\| > \delta$, because the sum of the first three terms is in $V_n$ while the fourth is orthogonal to $V_n$. Hence the sequence $\{x_n\}$ has no point of accumulation, denying the compactness of $K$.

Let $\mathcal{K}$ be the closure of $U_1^* \cup V_n$. $\mathcal{K}$ is complete and separable, i.e. it has a countable orthonormal base, the set $\{ e_n \}$ in fact. Further, $\mathcal{K} \supseteq K$. For, if $y \in K$, $y = y^n + u^n$, with $y^n \in V_n \subseteq \mathcal{K}$, and $\|u^n\| \leq a_{n+1}$ which becomes arbitrarily small with increasing $n$. Hence $y$ is arbitrarily close to $U_1^* \cup V_n$, the closure of which is $\mathcal{K}$. Thus, for any $x \in K$, $x = \sum_i x_i e_i$, and $\| \sum_i x_i e_i \| = (\sum_i |x_i|^2)^{1/2} \leq a_n$ for all $n$. We also observe, by assumption (2) of the opening of this proof, that $a_1 \leq 1$.

Now let $b_n = (2a_n)^{1/2}$. Clearly $b_n \geq (a_n)^{1/2} \geq a_n$, since $a_n \leq 1$. Also, $b_n \geq b_{n+1}$ for all $n$, and $\lim_n b_n = 0$. We construct $T$ as follows: Let $Te_i = b_i e_i$ for all $i$, and extend $T$ by linearity and continuity to all of $\mathcal{K}$. For $x \in \mathcal{K}$, define $Tx = \theta$, and again $T$ may be extended by linearity, this time to all of $H$.

To show $T$ is a compact transformation, it will suffice to apply
Lemma 1 to the set Cl(T(S)). Let $\varepsilon > 0$ be given, and choose $N$ such that $b_N < \varepsilon / 2$. Now if $y = \sum_{i=1}^{\infty} y_i e_i \in \text{Cl}(T(S))$, there exists some $x = \sum_{i=1}^{\infty} x_i e_i \in T(S)$ such that $\|x - y\| < \varepsilon / 2$. Further, there exists some $z = \sum_{i=1}^{\infty} z_i e_i \in S$ such that $Tz = x$, i.e. $\sum_{i=1}^{\infty} z_i b_i e_i = \sum_{i=1}^{\infty} x_i e_i$. Then $\sum_{i=1}^{n} (y_i - x_i) e_i + \sum_{i=1}^{\infty} x_i e_i \| \leq \| y - x \| + \| \sum_{i=1}^{\infty} b_i z_i e_i \| < \varepsilon / 2 + b_N \| z \| < \varepsilon$. Thus the criterion of Lemma 1 is fulfilled.

It only remains to be shown that $T(S) \supseteq K$. To this end, let $x = \sum_{i=1}^{\infty} x_i e_i \in K$. Then if $z = \sum_{i=1}^{\infty} (x_i / b_i) e_i, Tz = x$. To show $z \in S$ completes the proof. We shall show $\|z\|^2 \leq 1$ by showing that

$$\sum_{i=1}^{n} \left( \frac{|x_i|^2}{b_i^2} \right) \leq 1$$

for all $n$. If we set $R_n = \sum_{i=1}^{\infty} |x_i|^2$, and observe that $R_n \leq a_n^2$, then

$$\sum_{i=1}^{n} \left( \frac{|x_i|^2}{b_i^2} \right) = \sum_{i=1}^{n} \frac{R_i - R_{i+1}}{2a_i}$$

$$= \frac{1}{2} \left[ \sum_{i=1}^{n} \frac{R_{i+1}}{a_{i+1}} - \sum_{i=1}^{n} \frac{R_i}{a_i} \right]$$

$$= \frac{1}{2} \left[ \left( \frac{R_1}{a_1} - \frac{R_{n+1}}{a_n} \right) + \sum_{i=1}^{n-1} \frac{R_{i+1}}{a_{i+1}} \left( \frac{1}{a_{i+1}} - \frac{1}{a_i} \right) \right]$$

$$\leq \frac{1}{2} \left[ a_1 + \sum_{i=1}^{n-1} \frac{a_i - a_{i+1}}{a_i} \left( \frac{a_i}{a_{i+1}} \right) \right]$$

$$= \frac{1}{2} \left[ a_1 + \sum_{i=1}^{n-1} \frac{a_{i+1} - a_i}{a_i} \right]$$

$$\leq \frac{1}{2} \left[ a_1 + \sum_{i=1}^{n-1} (a_i - a_{i+1}) \right]$$

$$= \frac{1}{2} \left[ a_1 + a_1 - a_n \right]$$

$$\leq a_1 \leq 1.$$  

Q.E.D.

3. Alternate definition of the $k$-topology. Let $C(H, H)$ denote the class of all compact linear transformations, and let the $c$-topology be defined as follows: The typical neighborhood of $\theta$, $V(\theta) = \{ x \in H \mid \| T_i x \| \leq 1, T_i \in C(H, H), i = 1, 2, \cdots, n \}$. A basis of $c$-neighborhoods of $\theta$ is obtained by letting the finite sets $\{ T_i \}$ run through the class of all finite subsets of $C(H, H)$. That a locally convex topology is thus produced for $H$ is easily verified directly; indeed the $c$-topology is nothing but the point-open topology on $H$, when $H$ is regarded as a space of functions from $C(H, H)$ to $H$, under the rule $x : T \to Tx$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Theorem. The topologies $c$ and $k$ are identical.

Proof. Let $V(\theta)$ be any $k$-neighborhood: $V(\theta) = \{x \in H \mid \langle x, y \rangle \leq 1 \text{ for all } y \in K, K \text{ compact}\}$. Let $T \in C(H, H)$ be chosen (according to Lemma 1) so that $T(S) \supseteq K$, and let $T'$ be the adjoint of $T$. As is well known, $T' \in C(H, H)$ also. If $W$ is the $c$-neighborhood $\{x \in H \mid \|T'x\| \leq 1\}$, then $W \subset V$, for let $x \in W$. Then $\|T'x\| \leq 1$, and if $u \in S$, $\langle u, T'x \rangle \leq 1$. Thus $\langle Tu, x \rangle \leq 1$ for all $u \in S$, and a fortiori, $\langle y, x \rangle \leq 1$ for all $y \in K$. Conversely, let $W(\theta)$ be a $c$-neighborhood: $W = \{x \in H \mid \|T_i x\| \leq 1, T_i \in C(H, H), i = 1, 2, \ldots, n\}$. Then for all $u \in S$, $\langle u, T_i x \rangle \leq 1$, and $\langle T_i' u, x \rangle \leq 1$, for each $i$. The $T_i'$ are all in $C(H, H)$. Let $K = \bigcup_{i=1}^{n} \text{Cl}(T_i'(S))$. Then the $k$-neighborhood $V(\theta) = \{x \in H \mid \langle y, x \rangle \leq 1 \text{ for all } y \in K\}$ clearly satisfies $V(\theta) \subseteq W(\theta)$. Q.E.D.

The definitions for the $k$- and $c$-topologies can be generalized considerably, but even for complete normed spaces having a Schauder base it is not possible to prove Lemma 2 by the device used above. However, a statement somewhat weaker than Lemma 2 would suffice: that given a compact set $K$, there exist a finite number of compact transformations $T_i$ such that the union of $T_i(S)$ contain $K$.

Bibliography


University of Michigan