STRUCT IDEALS[1]

A. D. WALLACE

We shall be concerned with a theory initiated in the main by Nachbin [4] and developed and expanded by Ward [8] and [9]. Earlier fragmentary results are cited in the papers of Ward. This work exemplifies a trend in topological algebra—the transition from the algebraic operations to induced relations. Our Ursätze are to be found among the propositions established in recent years by Clifford, Faucett, Green, Koch and others (see the bibliography in [7]).

Let X be a *Hausdorff* space and let L be a subset of $X \times X$. Define $p, q\colon X \times X \to X$ by $p(x_1, x_2) = x_1$, $q(x_1, x_2) = x_2$, and let $\sigma\colon X \times X \to X \times X$ be defined by $\sigma(x_1, x_2) = (x_2, x_1)$. We shall write σL for $\sigma(L)$ and we note that $p\sigma = q$, $q\sigma = p$, $\sigma\sigma =$ the identity. For $A \subset X$ let

$$\begin{aligned} L(A) &= p((X \times A) \cap L) \\ &= \bigcup \{L(x) \mid x \in A\} \\ &= q((A \times X) \cap \sigma L). \end{aligned}$$

We term A an *L-ideal* if $A \neq \square$ and if $L(A) \subset A$. An equivalent condition is that $(X \times A) \cap L \subset A \times X$.

We state without proof some known and some easily proved results [1; 2; 4; 6; 9].

(i) $L(\bigcup\{A \mid A \in a\}) = \bigcup\{L(A) \mid A \in a\}$,

(ii) $L(\bigcap\{A \mid A \in a\}) \subset \bigcap\{L(A) \mid A \in a\}$,

(iii) *The union and intersection (if nonvoid) of L-ideals are L-ideals.*

(iv) $L(A) \cap B = p((B \times A) \cap L)$, $A \cap \sigma L(B) = q((B \times A) \cap L)$.

Hence the sets $L(A) \cap B$, $A \cap \sigma L(B)$, $(B \times A) \cap L$ are simultaneously void or nonvoid.

We define

$$L_0(A) = \bigcup\{x \cup L(x) \mid x \cup L(x) \subset A\}.$$

It will be noted that we do not distinguish between x and $\{x\}$ unless this is essential to the clarity of a statement.

(v) *If P and Q are compact sets in X, if L is closed, and if $(P \times Q) \cap L = \square$, then $(U \times V) \cap L = \square$ for some open sets $U \supset P$ and $V \supset Q$.*

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(vi) *If A and B are compact, if L is closed, and if $L(A)\cap B=\square$, then $L(U)\cap V=\square$ for some open sets $U\supset A$ and $V\supset B$. Hence $L(A)$ is closed when A is compact.*

(vii) *If X is compact, if U is open, and if L is closed, then*

$$L_0(U) = U\backslash\sigma L(X\backslash U)$$

is open. If L is transitive and if $L_0(A)\neq\square$, then $L_0(A)$ is the largest L-ideal contained in A.

We say that L is a *struct* on X if L is nonvoid, closed, and transitive. The word "struct" has been used by John Tukey in a different sense. We say that L is *continuous* if $L(A^*)\subset L(A)^*$ for each $A\subset X$. *This is a departure from the terminology of Nachbin and Ward.*

(viii) *If L is a struct on the compact space X, if A is an L-ideal, and if $x\in X\backslash A$, then $L_0(X\backslash x)$ is an open L-ideal including A and excluding x.*

THEOREM 1. *Let L be a struct on the compact space X, let A be an L-ideal, and let B be a closed set in X such that $B\cap(X\backslash A)\neq\square$ [$B\cap A\neq\square$]. Then among all L-ideals which satisfy with A the above conditions, there is a maximal [minimal] one and each such is open [closed].*

The proof of the unbracketed assertion can be made along the lines of a similar result in [**3**] and the bracketed assertion follows by an obvious duality.

COROLLARY. *Let X be compact and let L be a struct on X. Then there exist minimal L-ideals and each such is closed. If X properly includes an L-ideal, then there is a maximal proper L-ideal and each such is open.*

We say that $a\in X$ is L-minimal [L-maximal] if $x\cup L(x)\subset a\cup L(a)$ [$a\cup L(a)\subset x\cup L(x)$] implies the *equality* of these sets.

For $a\in X$ let

$$L_a = \{x \mid x\cup L(x) = a\cup L(a)\}.$$

(ix) *Let L be a struct on X. If L_a meets an L-ideal it is contained in it. We have $L_a=a\cup L(a)$ if and only if a is L-minimal. If L is a struct on X, then L_a is closed.*

THEOREM 2. *Let L be transitive.*

(a) *If A is a maximal proper L-ideal and if $a\in X\backslash A$, then a is L-maximal and $A=X\backslash L_a$.*

(b) *If $L_a\neq X$ then $X\backslash L_a$ is a maximal proper L-ideal if, and only if, a is L-maximal.*

(c) *If A is a minimal L-ideal then each $a\in A$ is L-minimal and $L_a=A$ for each $a\in A$.*

(d) *L_a is a minimal L-ideal if, and only if, a is L-minimal.*

From Theorems 1 and 2 we obtain the well-known

COROLLARY. *If X is compact and if L is a struct on X, then both L-maximal and L-minimal elements exist.*

If $L \subset X \times X$ we define K_L to be the set of all *L-minimal* elements.

THEOREM 3. *If L is a struct on X and if $K_L \neq \square$, then K_L is an L-ideal and $K_L = \bigcup \{L_x | x \in K_L\}$. If also X is compact and L is continuous, then K_L is closed.*

PROOF. We prove only that $K = K_L$ is closed. Let $a \in K^* \backslash K$. We easily see from the corollary to Theorem 2 that $z \cup L(z) \subset L(a)$ for some $z \in K$ and, from (vi), that there is an open set V about a with $V^* \cap (z \cup L(z)) = \square$. Let $Q = L_0(X \backslash V^*)$, so that Q is an open L-ideal by (vii). If $W = V \cap K$, then $a \in W^*$. If $Q \cap L(W) = \square$ then also $Q \cap L(W)^* \supset Q \cap L(W^*)$ is null. Since $a \in W^*$ it follows that $L(a) \cap Q = \square$. Hence $(z \cup L(z)) \cap Q = \square$ contrary to the fact that $z \cup L(z)$ is an L-ideal of $X \backslash V^*$ and so is contained in $Q = L_0(X \backslash V^*)$, see (vii). We see that $Q \cap L(W) \neq \square$ so that $L(p) \cap Q \neq \square$ for some $p \in W = V \cap K$. From the definition of $K = K_L$ and from (ix) we have $L_p = p \cup L(p) \subset Q$. We have $p \in Q \cap V$ contrary to the fact $Q \cap V^* = \square$.

We say that L is *monotone* if $L(x)$ is connected for each $x \in X$. It is usual to say that X is a *continuum* if it is a compact connected Hausdorff space. Let us note that it is a consequence of results due to I. S. Krule (which will form a part of his dissertation) that, if L is a continuous struct on the continuum X, then $L(x) \neq \square$ for all $x \in X$.

An element $a \in X$ is *L-reflexive* if $a \in L(a)$.

THEOREM 4. *If X is a continuum and if L is a continuous monotone struct on X, then K_L is a continuum.*

PROOF. Let $K = K_L = A \cup B$ where A and B are disjoint nonvoid *closed* sets, Theorem 3. Let $U = L_0(X \backslash B)$ so that U is an open L-ideal by (vii) and (ix). That $U \neq \square$ follows from a result of I. S. Krule (asserting that each $x \in K$ is L-reflexive) together with the fact that L is monotone. Let $V = L_0(X \backslash A)$. Then $U \cap V = \square$. For let $x \in U \cap V$ so that $L(x) \subset U \cap V \subset X \backslash (A \cup B) = X \backslash K$. Now $x \in X \backslash K$ since otherwise $L(x) \subset K$, by Theorem 3. Thus $x \cup L(x)$ does not meet K contrary to the corollary to Theorem 2. Hence $U \cap V = \square$ and thus $U^* \cap V = \square$. Since X is a continuum let $p \in U^* \backslash U = F(U)$. Now

$L(U)\subset U$ and thus $L(U^*)\subset L(U)^*\subset U^*$ so that $p\cup L(p)\subset U^*$. Hence $p\cup L(p)\subset X\backslash V\subset X\backslash B$. However $p\in U^*\backslash U$ implies that $p\cup L(p)$ is not contained in $X\backslash B$.

THEOREM 5. *Let L be a continuous monotone struct on the continuum X, let $a\in X\backslash K_L$ and let C be the component of $X\backslash L_a$ containing K_L. Then $L_a=a\cup(C^*\backslash C)$ and if a is L-reflexive then $a\in C^*\backslash C$.*

PROOF. Let $K=K_L$ and note that $L_a\cap K=\square$ since otherwise $L_a\subset K$. Let $U=L_0(X\backslash L_a)$ so that U is an open L-ideal. For any $x\in X$ we know that $x\cup L(x)$ meets K. Now

$$K\cup L(U)=K\cup\bigcup\{L(x)\,|\,x\in U\}$$

is a union of connected sets all meeting the connected set K so that $K\cup L(U)$ is connected and $K\cup L(U)\subset U\subset X\backslash L_a$. Hence $L(U)\subset C$. If $U^*\subset X\backslash L_a$, then $U^*\subset U$ because U^* is an L-ideal (L is continuous) and U is the largest L-ideal contained in $X\backslash L_a$. This implies that U is open and closed contrary to the fact that X is connected. Hence $U^*\cap L_a\neq\square$ and thus $a\in L_a\subset U^*$. We have $L(a)\subset L(U^*)\subset L(U)^*\subset C^*$ since $L(U)\subset C$. Thus $L_a\subset a\cup L(a)\subset a\cup C^*$ and if a is reflexive, $L_a\subset C^*$. Now $C\cap L_a=\square$ so that $L_a\subset a\cup(C^*\backslash C)$. If $x\in C^*\backslash C$, then $x\in L_a$ because otherwise $C\cup x\subset X\backslash L_a$ and $C\cup x$ is connected so that $C\cup x\subset C$. Hence $L_a=a\cup(C^*\backslash C)$ and in case a is reflexive, $L_a=C^*\backslash C$.

COROLLARY. *Let L be a continuous monotone reflexive struct on the continuum X. If a is L-maximal and if $a\in X\backslash K_L$, then no subset of L_a cuts X.*

PROOF. By Theorem 2 we know that $X\backslash L_a$ is an L-ideal. Now $X\backslash L_a$ is connected since it is the same as $K_L\cup\{L(x)\,|\,x\in X\backslash L_a\}$. Hence $C=X\backslash L_a$, notation of the theorem. Thus $C^*=C\cup(C^*\backslash C)=(X\backslash L_a)\cup L_a=X$. In other words C is a dense connected set so that $X\backslash A$ is connected if $A\subset X\backslash C=L_a$.

From the proof of Theorem 5 we note that we have (cf. [1])

COROLLARY. *Let L be a continuous monotone struct on the continuum X and let U be an open set about K_L with $U\neq X$. If C is the component of U containing K_L, then for some $p\in U^*\backslash U$ we have $L(p)\subset C^*$.*

It is of interest to observe that (notation of the corollary) if L is also reflexive, then $L_0(U)^*$ intersects $U^*\backslash U$. We recall the classical result of Janiszewski that C^* intersects $U^*\backslash U$, to which the above observation bears an obvious analogy.

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