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ON COMPLETE LATTICES AND A PROBLEM OF BIRKHOFF AND FRINK[1]

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Two decomposition theorems for elements of complete lattices in terms of join prime or completely join prime elements are obtained (Theorems 1 and 2). The latter theorem gives a solution for a problem of Birkhoff and Frink on the relation between completely prime and completely join irreducible ideals. An interesting special case of this solution—for complemented lattices—is also noted (Theorem 3).

1. Definitions. An element a of a lattice L is called *join irreducible* if $a_1 \cup a_2 = a \rightarrow a_1$ or $a_2 = a$, and *join prime* if $a_1 \cup a_2 \geqq a \rightarrow a_1$ or $a_2 \geqq a$. Similarly a is called *completely* join irreducible if (for all existing joins $\bigcup_i a_i$) $\bigcup_i a_i = a \rightarrow$some $a_i = a$, and *completely* join prime if $\bigcup_i a_i \geqq a \rightarrow$some $a_i \geqq a$.

The definitions of the corresponding dual concepts (indicated in each case by using the prefix "meet" in the place of "join") are quite clear.

A lattice L will be said to be infinitely meet distributive if all (existing) meets $\bigcap_i a_i$ in it are distributive; the meet $\bigcap_i a_i$ is said to

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be distributive if for arbitrary element a there holds the equality: $a\cup(\bigcap_i a_i)=\bigcap_i(a\cup a_i)$.

The symbol $\succ$ will denote the usual covering relation so that $a\succ b$ means that $a>b$ and there is no element x with $a>x>b$.

2. Lemmas.

LEMMA 1. *A join prime element of L is join irreducible; conversely, if L is distributive, a join irreducible element is join prime.*

PROOF. The first part follows from the definitions. For the second part, let L be distributive and $a\in L$ be join irreducible. Then, if $a_1\cup a_2\geqq a$, $a=(a_1\cup a_2)\cap a=(a_1\cap a)\cup(a_2\cap a)$, so that, since a is join irreducible, $a=a_1\cap a$ or $a_2\cap a$, i.e., $a\leqq a_1$ or a_2, whence a is join prime.

LEMMA 2. *Every join prime element $p(\neq 0)$ of a complemented lattice is an atom.*[3]

PROOF. If $a<p$, and a'—a complement of a, then since $a\cup a'=1\geqq p$, $a\nleqq p$, and p is join prime, there results: $a'\geqq p$, whence $a'\geqq p>a$ so that $a=a\cap a'=0$, or p is an atom.

LEMMA 3.[4] *Let L be a complete, infinitely meet distributive lattice. Then if a is such that* (C_1) $a>b\rightarrow a\succ b_1\geqq b$ *for some b_1, a is a join of completely join irreducibles.*

PROOF. It can be clearly supposed that a itself is not completely join irreducible. Since 0 is completely join irreducible the join b of all completely join irreducibles $\leqq a$ exists, and $b\leqq a$.

If possible, assume that $b<a$. Then by (C_1) there exists a b_1 with $b\leqq b_1\prec a$. If $x<a$ always implied $x\leqq b_1$, then a itself would be completely join irreducible (for, $x_i<a\rightarrow x_i\leqq b_1\rightarrow\bigcup_i x_i\leqq b_1<a$). Hence there exists at least an x with $x<a$, $x\nleqq b_1$.

Now let $c=\bigcap_i x_i$ be the meet of all $x<a$, $x\nleqq b_1$; then $c\nleqq b_1$. For, $c=\bigcap_i x_i\leqq b_1$ would imply: $b_1=b_1\cup(\bigcap_i x_i)=\bigcap_i(b_1\cup x_i)$; and since $b_1\prec a$, $b_1\leqq b_1\cup x_i\leqq a$, and $x_i\nleqq b_1$, this gives $b_1\cup x_i=a$, so that $b_1=\bigcap a=a$—a contradiction (since $b_1\prec a$). Hence $c\nleqq b_1$, and so a fortiori $c\nleqq b$.

Next, if $y<c$, then $y<c<a$; again $y\leqq b_1$ (else y would have to be an "x_i" which is impossible, since $y<c\leqq x_i$). Hence $y\leqq c\cap b_1<c$ (since $c\nleqq b_1$). It follows that c is completely join irreducible.

Thus, c is a completely join irreducible element with $c\nleqq b$, $c\leqq a$—which contradicts the definition of b, whence $b=a$, and the proof is complete.

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[3] An element a of a lattice with 0 is called an atom if $a\neq 0$, and $x<a\rightarrow x=0$.

[4] For a related result see [2, p. 304, Theorem 7] (citations in square brackets refer to the bibliography).

LEMMA 4. *Let L be a lattice in which every element is a join of a finite number of join primes. Then the lattice L^* of ideals of L is infinitely meet distributive; in particular, any existing meet $\bigcap_i a_i$ in L is distributive.*

PROOF. For establishing the first assertion it is sufficient to prove the inequality: $\bigcap_i(A\cup B_i)\subseteq A\cup(\bigcap_i B_i)$ where A, B_i are ideals of L.

If $x=\bigcup_{j=1}^n p_j$ (the p_j's being join prime elements) is an arbitrary element of $\bigcap_i (A\cup B_i)$, then of course $x\in(A\cup B_i)$ (for each i) so that x has the form $x\leqq a_i\cup b_i(a_i\in A,\ b_i\in B_i)$. Since $p_j\leqq\bigcup_{j=1}^n p_j=x\leqq a_i\cup b_i$, and p_j is join prime, $p_j\leqq a_i$ or b_i, i.e., $p_j\in A$ or B_i. Hence $p_j\in A$, or $p_j\in$*every* B_i (and hence also$\in\bigcap_i B_i$). Thus (each) $p_j\in A\cup(\bigcap_i B_i)$ so that $x=\bigcup_{j=1}^n p_j\in A\cup(\bigcap_i B_i)$, which proves the stated inequality, and thereby the first assertion.

The second assertion follows from the first by noting that the isomorphism between elements and principal ideals in L preserves all existing meets of elements.

COROLLARY. *If L is complete and every element of L is a join of a finite number of join primes, then L is infinitely meet distributive.*

LEMMA 5. *If L is a complete lattice in which every element is a meet of completely meet primes, then L is infinitely meet distributive.*

PROOF. It suffices to prove the inequality $a\cup(\bigcap_i a_i)\geqq\bigcap_i(a\cup a_i)$, where $\bigcap_i a_i$ is any meet in L.

By hypothesis, $a\cup(\bigcap_i a_i)=\bigcap_j c_j$, where the c_j's are completely meet primes. Since (each) $c_j\geqq\bigcap_j c_j=a\cup(\bigcap_i a_i)\geqq\bigcap_i a_i$, and c_j is completely meet prime, $c_j\geqq$some a_i; also $c_j\geqq\bigcap_j c_j\geqq a$. Hence, $c_j\geqq a\cup a_i\geqq\bigcap_i(a\cup a_i)$, and therefore $a\cup(\bigcap_i a_i)=\bigcap_j c_j\geqq\bigcap_i(a\cup a_i)$. Q.E.D.

LEMMA 6. *Every ideal of a lattice L is a meet of completely meet irreducible ideals*[5] (*cf.* [2, p. 307, *Theorem* 11]).

PROOF. Let $\bigcap_i A_i$ be the meet of all completely meet irreducible ideals $A_i\supseteq A$ (so that $\bigcap_i A_i\supseteq A$). If $\bigcap_i A_i\supset A$, there exists some element a with $a\in\bigcap A_i$, $a\notin A$. By Zorn's lemma there is a maximal chain $C=(B_j)$ of ideals B_j such that $B_j\supseteq A$, $B_j\not\ni a$. The join $B=\bigcup_j B_j$ (which is simply the set union of the B_j's on account of their linear ordering) also satisfies the same conditions, and hence belongs to C, since C is maximal. Further, if an ideal $I\supset B$, I lies outside the chain C (since B is the highest element of C); also $I\ni a$, as otherwise I would have to belong to C by virtue of maximality of C. Hence the meet

[5] By a completely meet irreducible ideal of L is meant a completely meet irreducible element of L^* (the lattice of ideals).

$\bigcap_i I_i$ of all ideals $I_i \supset B$ contains a, and so $\neq B$. That is, B is completely meet irreducible.

Again since $B \supseteq A$, B must be one of the A_i's so that $B \supset \bigcap_i A_i \ni a$; on the other hand since $B \in C$, $B \not\ni a$. This contradiction proves $\bigcap_i A_i = A$.

3. The main theorems.

THEOREM 1. *Let L have the zero element and satisfy the ascending chain condition. Then every element of L is a join of a finite number of join primes if, and only if, L is infinitely meet distributive.*

PROOF. Let L be infinitely meet distributive. Since the ascending chain condition holds, it is clear that every element a satisfies condition C_1 (of Lemma 3). Hence by Lemma 3, $a = \bigcup_i c_i$, where c_i are (completely) join irreducible, and therefore also join prime (Lemma 1). Again, in view of the chain condition there exists a finite subset c_{i_k} $(k=1, \cdots, n)$ of the $\{c_i\}$ such that $\bigcup_{k=1}^n c_{i_k} = \bigcup_i c_i$ (for otherwise there would be in $\{c_i\}$ some infinite chain $c_1 < c_1 \cup c_2 < c_1 \cup c_2 \cup c_3 \cdots$, contradicting the chain condition). This completes the "if" part, while the "only if" part results from corollary to Lemma 4.

THEOREM 2. *The following assertions concerning a complete lattice L are all equivalent*:

(i) *Every element of L is a join of a finite number of join primes.*

(ii) *L is distributive and every element of L is a join of a finite number of join irreducible elements.*

(iii) *The lattice L^* of ideals of L is infinitely meet distributive.*

(iv) *Every completely meet irreducible ideal of L is a completely meet prime ideal.*[6]

PROOF. First, by Lemma 4, (i) implies that L^* (and hence also L) is distributive; whereas, in a distributive lattice every join irreducible element is join prime (Lemma 1). Hence (i) and (ii) are equivalent.

Next, (i) implies (iii) by Lemma 4. Again since in an infinitely meet distributive lattice every completely meet irreducible element is completely meet prime (cf. Lemma 1), (iii) implies (iv). Also (iv) implies (iii); for, Lemmas 6 and 4 give: Every ideal of L is a meet of completely meet prime ideals, whence (by Lemma 5) L^* is infinitely meet distributive.

The proof will be now completed by showing (iii) implies (i). Assume that (iii) holds, and let $A = P(a)$ be a principal ideal of L. Then $A \supset B \rightarrow A \succ B_1 \supseteq B$, for some ideal B_1. For, in the set S of ideals

[6] I.e., completely prime element of L^*.

which $\supseteq B$ and $\subset A$, there exists by Zorn's Lemma (cf. proof of Lemma 6) a maximal $B_1 \in S$, so that $A \succ B_1 \supseteq B$.

Now Lemma 3 can be applied to L^* (taking for a any principal ideal $A = P(a)$). It then results that $A = P(a) = \bigcup_i A_i$, where A_i are completely join irreducible ideals.

Since A_i is the join of all principal ideals contained in it, A_i itself must be a principal ideal (since A_i is completely join irreducible); and moreover, if $A = P(a_i)$ the element a_i is join irreducible, and so join prime (Lemma 1).

Thus, $A = P(a) = \bigcup_i P(a_i)$, whence $a \leqq a_1 \cup \cdots \cup a_n$ and so $a = a_1 \cup \cdots \cup a_n$ (since $a_i \leqq a$)—which completes the proof.

4. **Relation to a problem of Birkhoff and Frink.** In their paper [2] Birkhoff and Frink have raised (on p. 313) the problem of finding all complete lattices L in which (C_2) *every completely meet irreducible dual ideal is completely prime.*[7] Theorem 2 (or more correctly its dual form formulated for dual ideals) furnishes some information on such lattices.

If in the above problem the complete lattices L are further restricted to be complemented, then we obtain a surprisingly simple characterization for the lattices answering to the conditions of the problem: viz.,

THEOREM 3. *The complete, complemented lattices L satisfying the condition: every completely meet irreducible ideal is completely meet prime, are precisely the finite Boolean algebras.*

PROOF. That every finite Boolean algebra B satisfies the stated condition[8] follows from Lemma 1 and the known fact that the lattice of ideals of a Boolean algebra is distributive.

Conversely, let L be a complete, complemented lattice satisfying this condition. By Theorem 2 it results that (i) L is distributive, (ii) each element a is a join $a = \bigcup_{i=1}^{m} p_i$ of finite number of join prime elements p_i. Since L is complemented, by Lemma 2, p_i is either 0 or is an atom. Thus, every element ($\neq 0$) of L is a join of a finite number of atoms.

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[7] According to [2] a dual ideal A is completely prime if $\bigcup_i a_i \in A \rightarrow$ some $a_i \in A$. Now let A be completely prime dual ideal of a complete lattice L and $A \supseteq \bigcap_i A_i$. Then $A \supseteq$ some A_i; else there would exist $a_i \in A_i$, $a_i \notin A$, so that $\bigcup_i a_i \in \bigcap_i A \subseteq A$—contradicting the above definition. Hence for a complete lattice a completely prime dual ideal is the same as a completely meet prime dual ideal—in the sense of the present paper (cf. footnote 6).

[8] Note that since B is finite the adjectives "completely" in the condition lose significance.

Next, let $P=(p_i)$ denote the set of all join prime elements ($p_i=0$, or is an atom). If $1=\bigcup_{i=1}^m p_i$ ($p_i\neq 0$) and $p(\neq 0)$ are in P, then $0<p\leqq p\cap(\bigcup_{i=1}^m p_i)=\bigcup_{i=1}^m (p\cap p_i)$, whence $p\cap p_i\neq 0$ (for some $i=i_1$). Hence, since p, p_{i_1} are atoms, $p=p_{i_1}$. Thus, P is the finite set $\{0, p_1, \cdots, p_m\}$, and since each element of L is a join of elements from P, L itself is finite. Finally, since L is complemented (hypothesis) and distributive (proved), L is a Boolean algebra. (L is, in fact, the finite Boolean algebra of all subsets of P.) The proof is complete.

Incidentally, the above theorem (in its dual form) corrects the erroneous assertion made by Birkhoff and Frink (loc. cit.) that every complete, atomic Boolean algebra B satisfies the condition (C_2). This assertion holds only in the case B is finite.

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