# THE AVERAGE OF THE RECIPROCAL OF A FUNCTION

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1. **Introduction.** Let $f(x)$ be an integrable function defined on an interval $(a, b)$. Its average value is

$$A(f) = \frac{1}{b-a}\int_a^b f(x)dx.$$

There are circumstances [1; 2] in which one wishes to calculate not $A(f)$ but

$$B(f) = [A(f^{-1})]^{-1} = (b-a)\Big/\int_a^b \frac{dx}{f(x)},$$

although $A(f)$ is much easier to calculate. If $f(x)$ is constant, $A(f) = B(f)$, and it is reasonable to expect that $A(f)$ will be approximately equal to $B(f)$ if $f(x)$ does not vary too widely. We propose to determine here the extreme values of the ratio

$$I(f) = A(f)/B(f)$$

as $f$ varies over a special class of functions.

Suppose that $0<\alpha<\beta$, and that $\mathfrak{A}$ is the class of measurable functions $f(x)$ defined on $(a, b)$ for which $\alpha \leqq f(x) \leqq \beta$. It is a consequence of a result of Pólya and Szegö [3] that

$$1 \leqq I(f) \leqq (\alpha+\beta)^2/4\alpha\beta \tag{1}$$

when $f(x)$ is in $\mathfrak{A}$. If $\mathfrak{B}$ is the class of concave (i.e., arc lies above chord) monotone decreasing functions $f(x)$ in $\mathfrak{A}$ which assume the values $\alpha$ and $\beta$, then we shall prove the better result that

$$1 \leqq I(f) \leqq \beta\left[\frac{\beta\ln(\beta/\alpha)}{\beta-\alpha} - \frac{\beta+\alpha}{2\beta}\right]^2 \Big/ 2(\beta-\alpha)\left[\frac{\beta\ln(\beta/\alpha)}{\beta-\alpha} - 1\right]. \tag{2}$$

These same bounds apply if $f(x)$ is a concave, monotone function in $\mathfrak{A}$, since the transformation $x' = a+b-x$ converts increasing functions into decreasing functions without altering concavity or bounds, and since the right-hand side of (2) is a strictly increasing function of $\beta/\alpha$.

2. **Existence of a maximizing function in $\mathfrak{B}$.** Our proof of the in-

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equality (2) consists of a demonstration of the existence of a function $f_0(x)$ in $\mathfrak{B}$ for which $I(f)$ attains its least upper bound on $\mathfrak{B}$, followed by the deduction of various properties which any such maximizing function must possess. There will be precisely one function in $\mathfrak{B}$ possessing these properties and so it must be the maximizing function.

THEOREM 1. *There exists a function $f_0(x)$ in $\mathfrak{B}$ for which $I(f)$ is a maximum.*

We begin by observing that when $f$ is concave, there exists a function $f'(x)$ which is decreasing and integrable on $(a, b)$ such that

$$f(x) = f(a+) + \int_a^x f'(t)dt, \qquad a < x < b.$$

The set of discontinuities of $f'(x)$ on the open interval $(a, b)$ is a finite or denumerable set $E(f)$ and so $f'(x)$ is the derivative of $f(x)$ except on $E(f)$ and possibly at $a$ and $b$. In addition when $f(x)$ is in $\mathfrak{B}$, $f'(x) \leqq 0$. Moreover $f(x)$ is continuous when $a \leqq x < b$, and $f(b-) \geqq f(b) = \alpha$.

LEMMA 1. *If $f(x)$ is in $\mathfrak{B}$ and $c = b - M^{-1}(\beta - \alpha) > a$, then $f'(x) \geqq -M$ when $a \leqq x \leqq c$.*

For suppose there were a point $y$ such that $a \leqq y \leqq c$, $f'(y) < -M$. Then, since $f'(x)$ is decreasing, $f'(t) < -M$ when $y \leqq t \leqq b$. Hence

$$\beta \geqq f(y) = f(b-) - \int_y^b f'(t)dt > \alpha + (b - y)M \geqq \alpha + (b - c)M = \beta,$$

and this is impossible.

Let $\mu$ be the least upper bound of $I(f)$ on $\mathfrak{B}$, and pick a sequence $f_n(x)$ of functions in $\mathfrak{B}$ for which $\mu = \lim I(f_n)$. These functions may be chosen as continuous. Pick a monotone increasing sequence of numbers $M_k$ such that

$$a < c_k = b - M_k^{-1}(\beta - \alpha) \to b.$$

According to Lemma 1, $0 \geqq f_n'(x) \geqq -M_k$ when $a \leqq x \leqq c_k$, and so the functions $f_n(x)$ are equicontinuous and uniformly bounded on the closed interval $(a, c_k)$. By Ascoli's theorem, there exists a subsequence of the sequence $f_n(x)$ which converges uniformly to a limit $g_1(x)$ on $(a, c_1)$, a subsequence of this subsequence which converges uniformly to a limit $g_2(x)$ on $(a, c_2)$, etc. It is clear that $g_{k+1}(x) = g_k(x)$ when $a \leqq x \leqq c_k$ and that if we define $f_0(x)$ as $\lim g_k(x)$ when $a \leqq x < b$, then we can by the diagonal process select a subsequence of $f_n(x)$ which converges uniformly to $f_0(x)$ on every closed interval $(a, c)$ for which $c<b$. If we define $f_0(b)$ to be $\alpha$, then it is easy to see that $f_0(x)$ is in $\mathfrak{B}$ and that

$$\mu = \lim I(f_n) = I(f_0).$$

Hence Theorem 1 is true.

Let $\mathfrak{M}$ be the class of maximizing functions for $I(f)$ in $\mathfrak{B}$; then the result of Theorem 1 is that $\mathfrak{M}$ is not void.

3. **A useful identity.** Most of our remaining analysis will depend in one way or another on the following result, the proof of which is obvious.

LEMMA 2. *If $f_0(x)$ and $f(x)$ are in $\mathfrak{B}$, if $\eta(x)=f(x)-f_0(x)$, and if $A(\eta)\neq 0$, then*

$$\frac{I(f) - I(f_0)}{A(\eta)} = A(f_0^{-1}) - \frac{A(f_0)A(\eta/f_0f)}{A(\eta)} - A(\eta/f_0f).$$

We shall use the lemma first to prove the following result.

LEMMA 3. *If $f_0(x)$ is in $\mathfrak{M}$, then $f_0(x)$ is continuous on the closed interval $(a, b)$.*

Since any function in $\mathfrak{B}$ is continuous when $a\leqq x<b$, it is sufficient to show that $f_0(x)$ is continuous when $x=b$. Since $f_0(b-)$ exists, suppose that $f_0(b-)=\alpha'>\alpha$. Choose a positive number $\delta$ for which $\alpha'-\delta>\alpha$. Define $\eta(x, \epsilon)$ as 0 when $a\leqq x\leqq b-\epsilon$ and when $x=b$, and so that $f(x, \epsilon)=f_0(x)+\eta(x, \epsilon)$ is linear on the open interval $(b-\epsilon, b)$ with limiting end values $f_0(b-\epsilon)$ and $\alpha'-\delta$. Hence

$$\alpha' - \delta - \beta \leqq \eta(x, \epsilon) < 0 \qquad (b - \epsilon < x < b).$$

Then $f(x, \epsilon)$ is in $\mathfrak{B}$ for sufficiently small $\epsilon$ and so

$$(I(f) - I(f_0))/A(\eta) \geqq 0.$$

From the first theorem of the mean for integrals,

$$A(\eta/f_0f) = A(\eta)/f_0(x^*)f(x^*),$$

in which $b-\epsilon<x^*<b$. Since $A(\eta)\to 0$, $f_0(x^*)\to\alpha'$, and $f(x^*)\to\alpha'-\delta$, it follows from Lemma 2 that

$$A(f_0^{-1}) - A(f_0)/\alpha'(\alpha' - \delta) \geqq 0,$$

$$\alpha'(\alpha' - \delta) \geqq A(f_0)B(f_0).$$

On the other hand, since $f_0(x)$ is decreasing, $f_0(x)\geqq\alpha'$ when $a\leqq x<b$,

and so

$$A(f_0)B(f_0) \geqq \alpha'^2.$$

These last two inequalities are incompatible when $\delta > 0$, and so Lemma 3 must be true.

**4. The behavior of the derivative $f_0'(x)$ of a maximizing function.** We know that $f_0'(x)$ is a decreasing, nonpositive function and hence the set $E(f_0)$ of its discontinuities on the open set $(a, b)$ is at most denumerable. Our next result is the following lemma.

LEMMA 4. *If $f_0(x)$ is in $\mathfrak{M}$ then $f_0'(x)$ is constant on any interval of continuity of $f_0'(x)$.*

If Lemma 4 is false, there exists an interval $(y, z)$ such that $f_0'(x)$ is continuous when $y < x < z$ and $f_0'(y+) > f_0'(z-)$. Therefore, there exists a decreasing sequence $x_n$ for which $x_n \to y$, $f_0'(x_{n+1}) > f_0'(x_n)$. Define $\eta_n(x)$ as 0 when $x$ is not on the interval $(y, x_n)$ and so that $f_n(x) = f_0(x) + \eta_n(x)$ is linear on the interval $(y, x_n)$ with end values $f_0(y)$ and $f_0(x_n)$. If $\eta_n(x)$ were identically zero, $f_0(x)$ would be linear on the interval $(y, x_n)$, and so $f_0'(x_{n+1}) = f_0'(x_n)$. Hence $\eta_n(x)$ does not vanish identically. Moreover, $\eta_n(x) < 0$ when $y < x < x_{n+1}$ since it is convex on the interval $(y, x_n)$ and vanishes at the endpoints of that interval. The function $f_n(x)$ is in $\mathfrak{B}$ and so

$$\lim_{n\to\infty} \frac{I(f_n) - I(f_0)}{A(\eta_n)} \geqq 0.$$

Moreover,

$$A(\eta_n/f_0f_n) = \frac{A(\eta_n)}{f_0(x_n^*)f_n(x_n^*)},$$

in which $y < x_n^* < x_n$, $A(\eta_n) \to 0$, and so

$$\begin{aligned} &A(f_0^{-1}) - A(f_0)/[f_0(y)]^2 \geqq 0, \\ &[f_0(y)]^2 \geqq A(f_0)B(f_0) \equiv C^2. \end{aligned} \tag{3}$$

Now define $\zeta_n(x)$ as 0 when $x$ is not in the interval $(y, x_n)$, and so that $g_n(x) = f_0(x) + \zeta_n(x)$ is linear on an interval $(y, \gamma_n)$ with slope $f_0'(y+)$, is linear on the interval $(\gamma_n, x_n)$ with slope $f_0'(x_n)$, and is continuous on $(a, b)$. Then $y < \gamma_n < x_n$ and $\zeta_n(\gamma_n) > 0$, since $f_0(x)$ is not linear on the interval $(y, x_n)$. Moreover $\zeta_n(x) \geqq 0$ since $f_0(x)$ is concave and hence $\zeta_n(x) > 0$ when $y < x < x_n$. The function $g_n(x)$ is in $\mathfrak{B}$ and so

$$\lim_{n\to\infty} \frac{I(g_n) - I(f_0)}{A(\zeta_n)} \leqq 0.$$

This limit is evaluated exactly as in the preceding paragraph, and we deduce that

$$[f_0(y)]^2 \leqq A(f_0)B(f_0) \equiv C^2. \tag{4}$$

We conclude from the inequalities (3) and (4) that $f_0(y) = C$. It is clear that we could use the same arguments at $z$ and deduce also that $f_0(z) = C$. Since $f_0(x)$ is decreasing, we must then have that $f_0(x) = C$ when $y \leqq x \leqq z$, and so $f_0'(x) = 0 = f_0'(y+) = f_0'(z-)$. From this contradiction we infer the truth of Lemma 4.

5. **The set $E(f_0)$ of discontinuities of $f_0'(x)$.** We are going to show ultimately that the set $E(f_0)$ consists of precisely one point for any function $f_0(x)$ in $\mathfrak{M}$. The first step in this demonstration is the following result.

LEMMA 5. *If $f_0(x)$ is in $\mathfrak{M}$, then the set $E(f_0)$ is not void.*

If $E(f_0)$ is void, then $(a, b)$ is an interval of continuity of $f_0'(x)$ and so $f_0'(x)$ is constant on $(a, b)$. This constant must be

$$(\beta - \alpha)/(a - b),$$

and so

$$f_0(x) = \beta - \frac{(\beta - \alpha)(x - a)}{(b - a)},$$

$$I(f_0) = \frac{(\alpha + \beta) \ln (\beta/\alpha)}{2(\beta - \alpha)}.$$

Let us define $f(x, \epsilon)$ so that

$$f(x, \epsilon) = \begin{cases} \beta & (a \leqq x \leqq a + \epsilon), \\ \beta - (\beta - \alpha)(x - a - \epsilon)/(b - a - \epsilon) & (a + \epsilon \leqq x \leqq b). \end{cases}$$

Then $f(x, \epsilon)$ is in $\mathfrak{B}$ and

$$\begin{aligned} I(f) = I(f_0) &+ \frac{\epsilon}{2(b - a)}\left(\frac{1 + \xi}{\xi} - \frac{2 \ln \xi}{\xi - 1}\right) \\ &+ \frac{\epsilon^2}{2(b - a)^2}\left(\frac{\xi - 1}{\xi} - \ln \xi\right), \end{aligned} \tag{5}$$

in which $\xi = \beta/\alpha > 1$. It is easy to verify that the coefficient of

$\epsilon/2(b-a)$ is positive when $\xi>1$ and hence $I(f)>I(f_0)$ for sufficiently small positive $\epsilon$. From this contradiction we infer the truth of Lemma 5.

LEMMA 6. *If $f_0(x)$ is in $\mathfrak{M}$ and if $y$ is any point in the set $E(f_0)$ then*

$$f_0(y) \geqq C = \left\{A(f_0)B(f_0)\right\}^{1/2}. \tag{6}$$

We define $\eta(x, \epsilon)$ as 0 when $|x-y| \geqq \epsilon$ and so that $f(x, \epsilon)=f_0(x)+\eta(x, \epsilon)$ is linear on the interval $(y-\epsilon, y+\epsilon)$ and continuous on $(a, b)$. Then $\eta(x, \epsilon)<0$ when $y-\epsilon<x<y+\epsilon$, $f(x, \epsilon)$ is in $\mathfrak{B}$, and so

$$\lim_{\epsilon\to 0} \frac{I(f)-I(f_0)}{A(\eta)} \geqq 0.$$

The limit may be evaluated exactly as in the proof of Lemma 4, and leads immediately to the inequality (6).

LEMMA 7. *If $f_0(x)$ is in $\mathfrak{M}$ and if $y$ is any point in the set $E(f_0)$ for which there exists another point $z$ in $E(f_0)$ such that $z<y$ and the interval $(z, y)$ contains no other points of $E(f_0)$, then*

$$f_0(y) = C = \left\{A(f_0)B(f_0)\right\}^{1/2}.$$

Choose $\lambda$ so that $f_0'(z+)<\lambda<f_0'(z-)$. Define $\eta(x, \lambda)$ as 0 when $a\leqq x\leqq z$ and when $y\leqq x\leqq b$ and so that $f(x, \lambda)=f_0(x)+\eta(x, \lambda)$ is linear on an interval $(z, \gamma)$ with slope $\lambda$, is linear on the interval $(\gamma, y)$ with slope $f_0'(y+)$, and is continuous on $(a, b)$. Since $f_0(x)$ is linear on the interval $(z, y)$ with slope $f_0'(z+)$, we have that $z<\gamma<y$ and that $\eta(x, \lambda)>0$ when $z<x<y$. Hence

$$\limsup_{\lambda\to f_0'(z+)} \frac{I(f)-I(f_0)}{A(\eta)} \leqq 0.$$

Since $\eta(x, \lambda)\geqq 0$ and both $f_0$ and $f$ are decreasing

$$-A(\eta/f_0f) \geqq -A(\eta)/[f_0(y)]^2.$$

Since $A(\eta)\to 0$, we deduce that

$$0 \geqq A(f_0^{-1}) - A(f_0)/[f_0(y)]^2,$$
$$f_0(y) \leqq C.$$

Since Lemma 6 holds, we conclude that Lemma 7 is true.

LEMMA 8. *If $f_0(x)$ maximizes $I(f)$ on $\mathfrak{B}$, and if $y$ is any point in the set $E(f_0)$ for which there exists an increasing sequence $z_n$ of points of $E(f_0)$ which converge to $y$, then $f_0(y)=C=\left\{A(f_0)B(f_0)\right\}^{1/2}$.*

Define $\eta_n(x)$ as 0 when $a \leqq x \leqq z_n$ and $y \leqq x \leqq b$, and so that $f_n(x) = f_0(x) + \eta_n(x)$ is linear on an interval $(z_n, \gamma_n)$ with slope $f_0'(z_n -)$, is linear on the interval $(\gamma_n, y)$ with slope $f_0'(y+)$, and is continuous on $(a, b)$. Then $z_n < \gamma_n < y$ and $\eta_n(x) > 0$ when $z_n < x < y$ since a corner $z_{n+1}$ occurs on the open interval $(z_n, y)$. Hence

$$A(f_0^{-1}) - \frac{A(f_0)}{[f_0(y)]^2} = \lim_{n\to\infty} \frac{I(f_n) - I(f_0)}{A(\eta_n)} \leqq 0,$$

$$f_0(y) \leqq C.$$

Since Lemma 6 holds, we conclude that Lemma 8 is true.

LEMMA 9. *If $f_0(x)$ is in $\mathfrak{M}$, then the set $E(f_0)$ has at most two points.*

Suppose on the contrary that $E(f_0)$ has three distinct points $u < v < w$. Then either Lemma 7 or Lemma 8 applies to the points $v$ and $w$ and so $f_0(v) = f_0(w) = C$. Since $f_0(x)$ is decreasing, $f_0(x) = C$ when $v < x < w$, $f_0'(v+) = 0$. On the other hand, $f_0'(x)$ is a nonpositive decreasing function and $f_0'(v+) < f_0'(v-) \leqq 0$. From this contradiction we infer that Lemma 9 is true.

LEMMA 10. *If $f_0(x)$ is in $\mathfrak{M}$, and if the set $E(f_0)$ has exactly two points $y < z$, then $f_0(y) = \beta$, $f_0(z) = C$. If $E$ has exactly one point $y$, then either $f_0(y) = \beta$ or $f_0(y) = C = \{A(f_0)B(f_0)\}^{1/2}$.*

Suppose in either case that $f_0(y) < \beta$. Then define $\eta(x, \epsilon)$ as 0 when $x \geqq y$ and so that $f(x, \epsilon) = f_0(x) + \eta(x, \epsilon)$ is linear on the interval $(y - \epsilon, y)$ with slope $f_0'(y+)$, is linear on the interval $(a, y - \epsilon)$, assumes the value $\beta$ when $x = a$, and is continuous on $(a, b)$. Then $\eta(x, \epsilon) > 0$ when $0 < x < y$, $A(\eta) \to 0$ with $\epsilon$, and $f(x, \epsilon)$ is in $\mathfrak{B}$ for sufficiently small positive $\epsilon$. Hence

$$-A(\eta/f_0 f) \geqq -A(\eta)/[f_0(y)]^2.$$

Since $A(\eta) \to 0$,

$$0 \geqq \limsup \frac{I(f) - I(f_0)}{A(\eta)} \geqq A(f_0^{-1}) - \frac{A(f_0)}{[f_0(y)]^2},$$

and so $f_0(y) \leqq C$. Since Lemma 6 holds, $f_0(y) = C$. This is sufficient to prove the second sentence of Lemma 10. If $E(f_0)$ has another point $z > y$, then $f_0(z) = C$ also since Lemma 7 holds. Hence $f_0(x) = C$ when $y \leqq x \leqq z$, $f_0'(y+) = 0$, and this is impossible. From this contradiction we infer the truth of the first sentence of Lemma 10.

LEMMA 11. *If $f_0(x)$ is in $\mathfrak{M}$, then the set $E(f_0)$ has exactly one point.*

Suppose on the contrary that $E(f_0)$ has exactly two distinct points $y<z$. Then from Lemma 10, $f_0(y)=\beta$, $f_0(z)=C$ and $f_0(x)$ is linear on the intervals $(a, y)$, $(y, z)$ and $(z, b)$. Define $f(x, \epsilon)$ to be $f_0(x)$ on the intervals $(a, y)$ and $(z, b)$, linear with slope $f_0'(z+)$ on the interval $(z-\epsilon, z)$, linear on the interval $(y, z-\epsilon)$, and continuous on $(a, b)$. Define

$$\Delta = \frac{(z-y)}{2(b-a)}\left[f_0'(z-) - f_0'(z+)\right].$$

Then $\Delta>0$, and

$$\begin{aligned} A(f) &= A(f_0') + \epsilon\Delta, \\ A(f^{-1}) &= A(f_0^{-1}) + \epsilon\Delta\Delta_1 + O(\epsilon^2) \end{aligned}$$

in which

$$\Delta_1 = \frac{2}{\beta - C}\left\{\frac{\ln(\beta/C)}{\beta - C} - \frac{1}{C}\right\}.$$

Hence

$$I(f) = I(f_0) + \epsilon\Delta\left\{A(f_0^{-1}) + A(f_0)\Delta_1\right\} + O(\epsilon^2),$$

and so the coefficient of $\epsilon\Delta$ must be nonpositive. On the other hand, since $C^2 = A(f_0)B(f_0)$,

$$A(f_0^{-1}) + A(f_0)\Delta_1 = \frac{A(f_0)}{C^2}\left\{1 + \frac{2}{\zeta - 1}\left(\frac{\ln\zeta}{\zeta - 1} - 1\right)\right\},$$

in which $\zeta = \beta/C > 1$. The quantity within the braces is always positive when $\zeta>1$, and from this contradiction we deduce the truth of Lemma 11.

Lemma 12. *If $f_0(x)$ is in $\mathfrak{M}$ the value $f_0(y)$ at the unique discontinuity $y$ of $f_0'(x)$ is $\beta$.*

Suppose on the contrary that $f_0(y)<\beta$. According to Lemma 10, $f_0(y)=C$. Let $f(x, \epsilon)$ be defined as in the proof of Lemma 10. Then

$$\begin{aligned} A(f) &= A(f_0) + \epsilon\Delta_2, \\ A(f^{-1}) &= A(f_0^{-1}) + \epsilon\Delta_2\Delta_1 + O(\epsilon^2), \end{aligned}$$

in which

$$\Delta_2 = \frac{(y-a)}{2(b-a)}\left[f_0'(y-) - f_0'(y+)\right] > 0.$$

Hence

$$I(f) = I(f_0) + \epsilon\Delta_2\{A(f_0^{-1}) + A(f_0)\Delta_1\} + O(\epsilon^2).$$

Since the coefficient of $\epsilon\Delta_2$ is the same quantity encountered in the proof of Lemma 11, it is positive and so $I(f) > I(f_0)$ for sufficiently small positive $\epsilon$. From this contradiction we infer the truth of Lemma 12.

6. **The maximizing function** $f_0(x)$. We are now in a position to prove our principal result.

THEOREM 2. *If $f(x)$ is in $\mathfrak{B}$, then*

$$1 \leqq I(f) \leqq I(f_0),$$

*in which*

$$f_0(x) = \begin{cases} \beta & (a \leqq x \leqq y), \\ \beta - (\beta - \alpha)(x - y)/(b - y) & (y \leqq x \leqq b), \end{cases}$$

$$y = a + \frac{(b - a)((1 + \xi)/\xi - 2 \ln \xi/(\xi - 1))}{2(\ln \xi - (\xi - 1)/\xi)}.$$

As a consequence of the preceding lemmas, we know that any maximizing function $f_0(x)$ must be of the form described in Theorem 2 for some value $y$. If $y = a + \epsilon$ we have already $I(f_0)$ in equation (5). This equation may be written as

$$I(f_0) = L + Mu + Nu^2,$$

in which

$$L = \frac{(\xi + 1) \ln \xi}{2(\xi - 1)},$$

$$M = \frac{1}{2}\left(\frac{1 + \xi}{\xi} - \frac{2 \ln \xi}{\xi - 1}\right) > 0,$$

$$N = \frac{1}{2}\left(\frac{\xi - 1}{\xi} - \ln \xi\right) < 0,$$

$$u = (y - a)/(b - a),$$

$$\xi = \beta/\alpha.$$

This function of $u$ is concave and attains its maximum value when

$$u = -M/2N.$$

Since $M > 0$, $N < 0$ and $2N + M < 0$, this value of $u$ lies in the open interval (0, 1) and yields the value of $y$ specified in the theorem. The maximum value is $(4LN-M^2)/4N$, and this value is the one specified in the inequality (2).

Hence the only maximizing function possible is the one specified in Theorem 2. Since a maximizing function is known to exist, this function must be a maximizing function and in fact the only maximizing function.

A short table of values of the upper bounds in (1) and (2) as a function of the ratio $\beta/\alpha$ is given below.

| $\beta/\alpha$ | upper bound on $\mathfrak{A}$ | upper bound on $\mathfrak{B}$ |
|---|---|---|
| 1.0 | 1.0000 | 1.0000 |
| 1.5 | 1.0417 | 1.0171 |
| 2.0 | 1.1250 | 1.0481 |
| 2.5 | 1.2250 | 1.0816 |
| 3.0 | 1.3333 | 1.1146 |

## References

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