

THE AVERAGE OF THE RECIPROCAL OF A FUNCTION

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1. **Introduction.** Let $f(x)$ be an integrable function defined on an interval (a, b) . Its average value is

$$A(f) = \frac{1}{b-a} \int_a^b f(x) dx.$$

There are circumstances [1; 2] in which one wishes to calculate not $A(f)$ but

$$B(f) = [A(f^{-1})]^{-1} = (b-a) \int_a^b \frac{dx}{f(x)},$$

although $A(f)$ is much easier to calculate. If $f(x)$ is constant, $A(f) = B(f)$, and it is reasonable to expect that $A(f)$ will be approximately equal to $B(f)$ if $f(x)$ does not vary too widely. We propose to determine here the extreme values of the ratio

$$I(f) = A(f)/B(f)$$

as f varies over a special class of functions.

Suppose that $0 < \alpha < \beta$, and that \mathfrak{A} is the class of measurable functions $f(x)$ defined on (a, b) for which $\alpha \leq f(x) \leq \beta$. It is a consequence of a result of Pólya and Szegő [3] that

$$(1) \quad 1 \leq I(f) \leq (\alpha + \beta)^2 / 4\alpha\beta$$

when $f(x)$ is in \mathfrak{A} . If \mathfrak{B} is the class of concave (i.e., arc lies above chord) monotone decreasing functions $f(x)$ in \mathfrak{A} which assume the values α and β , then we shall prove the better result that

$$(2) \quad 1 \leq I(f) \leq \beta \left[\frac{\beta \ln(\beta/\alpha)}{\beta - \alpha} - \frac{\beta + \alpha}{2\beta} \right]^2 \bigg/ 2(\beta - \alpha) \left[\frac{\beta \ln(\beta/\alpha)}{\beta - \alpha} - 1 \right].$$

These same bounds apply if $f(x)$ is a concave, monotone function in \mathfrak{A} , since the transformation $x' = a + b - x$ converts increasing functions into decreasing functions without altering concavity or bounds, and since the right-hand side of (2) is a strictly increasing function of β/α .

2. **Existence of a maximizing function in \mathfrak{B} .** Our proof of the in-

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equality (2) consists of a demonstration of the existence of a function $f_0(x)$ in \mathfrak{B} for which $I(f)$ attains its least upper bound on \mathfrak{B} , followed by the deduction of various properties which any such maximizing function must possess. There will be precisely one function in \mathfrak{B} possessing these properties and so it must be the maximizing function.

THEOREM 1. *There exists a function $f_0(x)$ in \mathfrak{B} for which $I(f)$ is a maximum.*

We begin by observing that when f is concave, there exists a function $f'(x)$ which is decreasing and integrable on (a, b) such that

$$f(x) = f(a+) + \int_a^x f'(t)dt, \quad a < x < b.$$

The set of discontinuities of $f'(x)$ on the open interval (a, b) is a finite or denumerable set $E(f)$ and so $f'(x)$ is the derivative of $f(x)$ except on $E(f)$ and possibly at a and b . In addition when $f(x)$ is in \mathfrak{B} , $f'(x) \leq 0$. Moreover $f(x)$ is continuous when $a \leq x < b$, and $f(b-) \geq f(b) = \alpha$.

LEMMA 1. *If $f(x)$ is in \mathfrak{B} and $c = b - M^{-1}(\beta - \alpha) > a$, then $f'(x) \geq -M$ when $a \leq x \leq c$.*

For suppose there were a point y such that $a \leq y \leq c$, $f'(y) < -M$. Then, since $f'(x)$ is decreasing, $f'(t) < -M$ when $y \leq t \leq b$. Hence

$$\beta \geq f(y) = f(b-) - \int_y^b f'(t)dt > \alpha + (b - y)M \geq \alpha + (b - c)M = \beta,$$

and this is impossible.

Let μ be the least upper bound of $I(f)$ on \mathfrak{B} , and pick a sequence $f_n(x)$ of functions in \mathfrak{B} for which $\mu = \lim I(f_n)$. These functions may be chosen as continuous. Pick a monotone increasing sequence of numbers M_k such that

$$a < c_k = b - M_k^{-1}(\beta - \alpha) \rightarrow b.$$

According to Lemma 1, $0 \geq f'_n(x) \geq -M_k$ when $a \leq x \leq c_k$, and so the functions $f_n(x)$ are equicontinuous and uniformly bounded on the closed interval (a, c_k) . By Ascoli's theorem, there exists a subsequence of the sequence $f_n(x)$ which converges uniformly to a limit $g_1(x)$ on (a, c_1) , a subsequence of this subsequence which converges uniformly to a limit $g_2(x)$ on (a, c_2) , etc. It is clear that $g_{k+1}(x) = g_k(x)$ when $a \leq x \leq c_k$ and that if we define $f_0(x)$ as $\lim g_k(x)$ when $a \leq x < b$, then we can by the diagonal process select a subsequence of $f_n(x)$ which

converges uniformly to $f_0(x)$ on every closed interval (a, c) for which $c < b$. If we define $f_0(b)$ to be α , then it is easy to see that $f_0(x)$ is in \mathfrak{B} and that

$$\mu = \lim I(f_n) = I(f_0).$$

Hence Theorem 1 is true.

Let \mathfrak{M} be the class of maximizing functions for $I(f)$ in \mathfrak{B} ; then the result of Theorem 1 is that \mathfrak{M} is not void.

3. A useful identity. Most of our remaining analysis will depend in one way or another on the following result, the proof of which is obvious.

LEMMA 2. *If $f_0(x)$ and $f(x)$ are in \mathfrak{B} , if $\eta(x) = f(x) - f_0(x)$, and if $A(\eta) \neq 0$, then*

$$\frac{I(f) - I(f_0)}{A(\eta)} = A(f_0^{-1}) - \frac{A(f_0)A(\eta/f_0f)}{A(\eta)} - A(\eta/f_0f).$$

We shall use the lemma first to prove the following result.

LEMMA 3. *If $f_0(x)$ is in \mathfrak{M} , then $f_0(x)$ is continuous on the closed interval (a, b) .*

Since any function in \mathfrak{B} is continuous when $a \leq x < b$, it is sufficient to show that $f_0(x)$ is continuous when $x = b$. Since $f_0(b-)$ exists, suppose that $f_0(b-) = \alpha' > \alpha$. Choose a positive number δ for which $\alpha' - \delta > \alpha$. Define $\eta(x, \epsilon)$ as 0 when $a \leq x \leq b - \epsilon$ and when $x = b$, and so that $f(x, \epsilon) = f_0(x) + \eta(x, \epsilon)$ is linear on the open interval $(b - \epsilon, b)$ with limiting end values $f_0(b - \epsilon)$ and $\alpha' - \delta$. Hence

$$\alpha' - \delta - \beta \leq \eta(x, \epsilon) < 0 \quad (b - \epsilon < x < b).$$

Then $f(x, \epsilon)$ is in \mathfrak{B} for sufficiently small ϵ and so

$$(I(f) - I(f_0))/A(\eta) \geq 0.$$

From the first theorem of the mean for integrals,

$$A(\eta/f_0f) = A(\eta)/f_0(x^*)f(x^*),$$

in which $b - \epsilon < x^* < b$. Since $A(\eta) \rightarrow 0$, $f_0(x^*) \rightarrow \alpha'$, and $f(x^*) \rightarrow \alpha' - \delta$, it follows from Lemma 2 that

$$\begin{aligned} A(f_0^{-1}) - A(f_0)/\alpha'(\alpha' - \delta) &\geq 0, \\ \alpha'(\alpha' - \delta) &\geq A(f_0)B(f_0). \end{aligned}$$

On the other hand, since $f_0(x)$ is decreasing, $f_0(x) \geq \alpha'$ when $a \leq x < b$,

and so

$$A(f_0)B(f_0) \geq \alpha'^2.$$

These last two inequalities are incompatible when $\delta > 0$, and so Lemma 3 must be true.

4. The behavior of the derivative $f'_0(x)$ of a maximizing function. We know that $f'_0(x)$ is a decreasing, nonpositive function and hence the set $E(f_0)$ of its discontinuities on the open set (a, b) is at most denumerable. Our next result is the following lemma.

LEMMA 4. *If $f_0(x)$ is in \mathfrak{M} then $f'_0(x)$ is constant on any interval of continuity of $f'_0(x)$.*

If Lemma 4 is false, there exists an interval (y, z) such that $f'_0(x)$ is continuous when $y < x < z$ and $f'_0(y+) > f'_0(z-)$. Therefore, there exists a decreasing sequence x_n for which $x_n \rightarrow y$, $f'_0(x_{n+1}) > f'_0(x_n)$. Define $\eta_n(x)$ as 0 when x is not on the interval (y, x_n) and so that $f_n(x) = f_0(x) + \eta_n(x)$ is linear on the interval (y, x_n) with end values $f_0(y)$ and $f_0(x_n)$. If $\eta_n(x)$ were identically zero, $f_0(x)$ would be linear on the interval (y, x_n) , and so $f'_0(x_{n+1}) = f'_0(x_n)$. Hence $\eta_n(x)$ does not vanish identically. Moreover, $\eta_n(x) < 0$ when $y < x < x_{n+1}$ since it is convex on the interval (y, x_n) and vanishes at the endpoints of that interval. The function $f_n(x)$ is in \mathfrak{B} and so

$$\lim_{n \rightarrow \infty} \frac{I(f_n) - I(f_0)}{A(\eta_n)} \geq 0.$$

Moreover,

$$A(\eta_n/f_0f_n) = \frac{A(\eta_n)}{f_0(x_n^*)f_n(x_n^*)},$$

in which $y < x_n^* < x_n$, $A(\eta_n) \rightarrow 0$, and so

$$(3) \quad \begin{aligned} A(f_0^{-1}) - A(f_0)/[f_0(y)]^2 &\geq 0, \\ [f_0(y)]^2 &\geq A(f_0)B(f_0) \equiv C^2. \end{aligned}$$

Now define $\zeta_n(x)$ as 0 when x is not in the interval (y, x_n) , and so that $g_n(x) = f_0(x) + \zeta_n(x)$ is linear on an interval (y, γ_n) with slope $f'_0(y+)$, is linear on the interval (γ_n, x_n) with slope $f'_0(x_n)$, and is continuous on (a, b) . Then $y < \gamma_n < x_n$ and $\zeta_n(\gamma_n) > 0$, since $f_0(x)$ is not linear on the interval (y, x_n) . Moreover $\zeta_n(x) \geq 0$ since $f_0(x)$ is concave and hence $\zeta_n(x) > 0$ when $y < x < x_n$. The function $g_n(x)$ is in \mathfrak{B} and so

$$\lim_{n \rightarrow \infty} \frac{I(g_n) - I(f_0)}{A(\zeta_n)} \leq 0.$$

This limit is evaluated exactly as in the preceding paragraph, and we deduce that

$$(4) \quad [f_0(y)]^2 \leq A(f_0)B(f_0) \equiv C^2.$$

We conclude from the inequalities (3) and (4) that $f_0(y) = C$. It is clear that we could use the same arguments at z and deduce also that $f_0(z) = C$. Since $f_0(x)$ is decreasing, we must then have that $f_0(x) = C$ when $y \leq x \leq z$, and so $f'_0(x) = 0 = f'_0(y+) = f'_0(z-)$. From this contradiction we infer the truth of Lemma 4.

5. The set $E(f_0)$ of discontinuities of $f'_0(x)$. We are going to show ultimately that the set $E(f_0)$ consists of precisely one point for any function $f_0(x)$ in \mathfrak{M} . The first step in this demonstration is the following result.

LEMMA 5. *If $f_0(x)$ is in \mathfrak{M} , then the set $E(f_0)$ is not void.*

If $E(f_0)$ is void, then (a, b) is an interval of continuity of $f'_0(x)$ and so $f'_0(x)$ is constant on (a, b) . This constant must be

$$(\beta - \alpha)/(a - b),$$

and so

$$f_0(x) = \beta - \frac{(\beta - \alpha)(x - a)}{(b - a)},$$

$$I(f_0) = \frac{(\alpha + \beta) \ln(\beta/\alpha)}{2(\beta - \alpha)}.$$

Let us define $f(x, \epsilon)$ so that

$$f(x, \epsilon) = \begin{cases} \beta & (a \leq x \leq a + \epsilon), \\ \beta - (\beta - \alpha)(x - a - \epsilon)/(b - a - \epsilon) & (a + \epsilon \leq x \leq b). \end{cases}$$

Then $f(x, \epsilon)$ is in \mathfrak{B} and

$$(5) \quad I(f) = I(f_0) + \frac{\epsilon}{2(b - a)} \left(\frac{1 + \xi}{\xi} - \frac{2 \ln \xi}{\xi - 1} \right) + \frac{\epsilon^2}{2(b - a)^2} \left(\frac{\xi - 1}{\xi} - \ln \xi \right),$$

in which $\xi = \beta/\alpha > 1$. It is easy to verify that the coefficient of

$\epsilon/2(b-a)$ is positive when $\xi > 1$ and hence $I(f) > I(f_0)$ for sufficiently small positive ϵ . From this contradiction we infer the truth of Lemma 5.

LEMMA 6. *If $f_0(x)$ is in \mathfrak{M} and if y is any point in the set $E(f_0)$ then*

$$(6) \quad f_0(y) \geq C = \{A(f_0)B(f_0)\}^{1/2}.$$

We define $\eta(x, \epsilon)$ as 0 when $|x-y| \geq \epsilon$ and so that $f(x, \epsilon) = f_0(x) + \eta(x, \epsilon)$ is linear on the interval $(y-\epsilon, y+\epsilon)$ and continuous on (a, b) . Then $\eta(x, \epsilon) < 0$ when $y-\epsilon < x < y+\epsilon$, $f(x, \epsilon)$ is in \mathfrak{B} , and so

$$\lim_{\epsilon \rightarrow 0} \frac{I(f) - I(f_0)}{A(\eta)} \geq 0.$$

The limit may be evaluated exactly as in the proof of Lemma 4, and leads immediately to the inequality (6).

LEMMA 7. *If $f_0(x)$ is in \mathfrak{M} and if y is any point in the set $E(f_0)$ for which there exists another point z in $E(f_0)$ such that $z < y$ and the interval (z, y) contains no other points of $E(f_0)$, then*

$$f_0(y) = C = \{A(f_0)B(f_0)\}^{1/2}.$$

Choose λ so that $f'_0(z+) < \lambda < f'_0(z-)$. Define $\eta(x, \lambda)$ as 0 when $a \leq x \leq z$ and when $y \leq x \leq b$ and so that $f(x, \lambda) = f_0(x) + \eta(x, \lambda)$ is linear on an interval (z, γ) with slope λ , is linear on the interval (γ, y) with slope $f'_0(y+)$, and is continuous on (a, b) . Since $f_0(x)$ is linear on the interval (z, y) with slope $f'_0(z+)$, we have that $z < \gamma < y$ and that $\eta(x, \lambda) > 0$ when $z < x < y$. Hence

$$\limsup_{\lambda \rightarrow f'_0(z+)} \frac{I(f) - I(f_0)}{A(\eta)} \leq 0.$$

Since $\eta(x, \lambda) \geq 0$ and both f_0 and f are decreasing

$$-A(\eta/f_0f) \geq -A(\eta)/[f_0(y)]^2.$$

Since $A(\eta) \rightarrow 0$, we deduce that

$$0 \geq A(f_0^{-1}) - A(f_0)/[f_0(y)]^2, \\ f_0(y) \leq C.$$

Since Lemma 6 holds, we conclude that Lemma 7 is true.

LEMMA 8. *If $f_0(x)$ maximizes $I(f)$ on \mathfrak{B} , and if y is any point in the set $E(f_0)$ for which there exists an increasing sequence z_n of points of $E(f_0)$ which converge to y , then $f_0(y) = C = \{A(f_0)B(f_0)\}^{1/2}$.*

Define $\eta_n(x)$ as 0 when $a \leq x \leq z_n$ and $y \leq x \leq b$, and so that $f_n(x) = f_0(x) + \eta_n(x)$ is linear on an interval (z_n, γ_n) with slope $f'_0(z_n -)$, is linear on the interval (γ_n, y) with slope $f'_0(y +)$, and is continuous on (a, b) . Then $z_n < \gamma_n < y$ and $\eta_n(x) > 0$ when $z_n < x < y$ since a corner z_{n+1} occurs on the open interval (z_n, y) . Hence

$$A(f_0^{-1}) - \frac{A(f_0)}{[f_0(y)]^2} = \lim_{n \rightarrow \infty} \frac{I(f_n) - I(f_0)}{A(\eta_n)} \leq 0, \quad f_0(y) \leq C.$$

Since Lemma 6 holds, we conclude that Lemma 8 is true.

LEMMA 9. *If $f_0(x)$ is in \mathfrak{M} , then the set $E(f_0)$ has at most two points.*

Suppose on the contrary that $E(f_0)$ has three distinct points $u < v < w$. Then either Lemma 7 or Lemma 8 applies to the points v and w and so $f_0(v) = f_0(w) = C$. Since $f_0(x)$ is decreasing, $f_0(x) = C$ when $v < x < w$, $f'_0(v +) = 0$. On the other hand, $f'_0(x)$ is a nonpositive decreasing function and $f'_0(v +) < f'_0(v -) \leq 0$. From this contradiction we infer that Lemma 9 is true.

LEMMA 10. *If $f_0(x)$ is in \mathfrak{M} , and if the set $E(f_0)$ has exactly two points $y < z$, then $f_0(y) = \beta$, $f_0(z) = C$. If E has exactly one point y , then either $f_0(y) = \beta$ or $f_0(y) = C = \{A(f_0)B(f_0)\}^{1/2}$.*

Suppose in either case that $f_0(y) < \beta$. Then define $\eta(x, \epsilon)$ as 0 when $x \geq y$ and so that $f(x, \epsilon) = f_0(x) + \eta(x, \epsilon)$ is linear on the interval $(y - \epsilon, y)$ with slope $f'_0(y +)$, is linear on the interval $(a, y - \epsilon)$, assumes the value β when $x = a$, and is continuous on (a, b) . Then $\eta(x, \epsilon) > 0$ when $0 < x < y$, $A(\eta) \rightarrow 0$ with ϵ , and $f(x, \epsilon)$ is in \mathfrak{B} for sufficiently small positive ϵ . Hence

$$-A(\eta/f_0f) \geq -A(\eta)/[f_0(y)]^2.$$

Since $A(\eta) \rightarrow 0$,

$$0 \geq \limsup \frac{I(f) - I(f_0)}{A(\eta)} \geq A(f_0^{-1}) - \frac{A(f_0)}{[f_0(y)]^2},$$

and so $f_0(y) \leq C$. Since Lemma 6 holds, $f_0(y) = C$. This is sufficient to prove the second sentence of Lemma 10. If $E(f_0)$ has another point $z > y$, then $f_0(z) = C$ also since Lemma 7 holds. Hence $f_0(x) = C$ when $y \leq x \leq z$, $f'_0(y +) = 0$, and this is impossible. From this contradiction we infer the truth of the first sentence of Lemma 10.

LEMMA 11. *If $f_0(x)$ is in \mathfrak{M} , then the set $E(f_0)$ has exactly one point.*

Suppose on the contrary that $E(f_0)$ has exactly two distinct points $y < z$. Then from Lemma 10, $f_0(y) = \beta$, $f_0(z) = C$ and $f_0(x)$ is linear on the intervals (a, y) , (y, z) and (z, b) . Define $f(x, \epsilon)$ to be $f_0(x)$ on the intervals (a, y) and (z, b) , linear with slope $f'_0(z+)$ on the interval $(z - \epsilon, z)$, linear on the interval $(y, z - \epsilon)$, and continuous on (a, b) . Define

$$\Delta = \frac{(z - y)}{2(b - a)} [f'_0(z-) - f'_0(z+)].$$

Then $\Delta > 0$, and

$$\begin{aligned} A(f) &= A(f'_0) + \epsilon\Delta, \\ A(f^{-1}) &= A(f_0^{-1}) + \epsilon\Delta\Delta_1 + O(\epsilon^2) \end{aligned}$$

in which

$$\Delta_1 = \frac{2}{\beta - C} \left\{ \frac{\ln(\beta/C)}{\beta - C} - \frac{1}{C} \right\}.$$

Hence

$$I(f) = I(f_0) + \epsilon\Delta \{ A(f_0^{-1}) + A(f_0)\Delta_1 \} + O(\epsilon^2),$$

and so the coefficient of $\epsilon\Delta$ must be nonpositive. On the other hand, since $C^2 = A(f_0)B(f_0)$,

$$A(f_0^{-1}) + A(f_0)\Delta_1 = \frac{A(f_0)}{C^2} \left\{ 1 + \frac{2}{\zeta - 1} \left(\frac{\ln \zeta}{\zeta - 1} - 1 \right) \right\},$$

in which $\zeta = \beta/C > 1$. The quantity within the braces is always positive when $\zeta > 1$, and from this contradiction we deduce the truth of Lemma 11.

LEMMA 12. *If $f_0(x)$ is in \mathfrak{M} the value $f_0(y)$ at the unique discontinuity y of $f'_0(x)$ is β .*

Suppose on the contrary that $f_0(y) < \beta$. According to Lemma 10, $f_0(y) = C$. Let $f(x, \epsilon)$ be defined as in the proof of Lemma 10. Then

$$\begin{aligned} A(f) &= A(f_0) + \epsilon\Delta_2, \\ A(f^{-1}) &= A(f_0^{-1}) + \epsilon\Delta_2\Delta_1 + O(\epsilon^2), \end{aligned}$$

in which

$$\Delta_2 = \frac{(y - a)}{2(b - a)} [f'_0(y-) - f'_0(y+)] > 0.$$

Hence

$$I(f) = I(f_0) + \epsilon \Delta_2 \{A(f_0^{-1}) + A(f_0) \Delta_1\} + O(\epsilon^2).$$

Since the coefficient of $\epsilon \Delta_2$ is the same quantity encountered in the proof of Lemma 11, it is positive and so $I(f) > I(f_0)$ for sufficiently small positive ϵ . From this contradiction we infer the truth of Lemma 12.

6. The maximizing function $f_0(x)$. We are now in a position to prove our principal result.

THEOREM 2. *If $f(x)$ is in \mathfrak{B} , then*

$$1 \leq I(f) \leq I(f_0),$$

in which

$$f_0(x) = \begin{cases} \beta & (a \leq x \leq y), \\ \beta - (\beta - a)(x - y)/(b - y) & (y \leq x \leq b), \end{cases}$$

$$y = a + \frac{(b - a)((1 + \xi)/\xi - 2 \ln \xi / (\xi - 1))}{2(\ln \xi - (\xi - 1)/\xi)}.$$

As a consequence of the preceding lemmas, we know that any maximizing function $f_0(x)$ must be of the form described in Theorem 2 for some value y . If $y = a + \epsilon$ we have already $I(f_0)$ in equation (5).

This equation may be written as

$$I(f_0) = L + Mu + Nu^2,$$

in which

$$L = \frac{(\xi + 1) \ln \xi}{2(\xi - 1)},$$

$$M = \frac{1}{2} \left(\frac{1 + \xi}{\xi} - \frac{2 \ln \xi}{\xi - 1} \right) > 0,$$

$$N = \frac{1}{2} \left(\frac{\xi - 1}{\xi} - \ln \xi \right) < 0,$$

$$u = (y - a)/(b - a),$$

$$\xi = \beta/\alpha.$$

This function of u is concave and attains its maximum value when

$$u = -M/2N.$$

Since $M > 0$, $N < 0$ and $2N + M < 0$, this value of u lies in the open

interval $(0, 1)$ and yields the value of y specified in the theorem. The maximum value is $(4LN - M^2)/4N$, and this value is the one specified in the inequality (2).

Hence the only maximizing function possible is the one specified in Theorem 2. Since a maximizing function is known to exist, this function must be a maximizing function and in fact the only maximizing function.

A short table of values of the upper bounds in (1) and (2) as a function of the ratio β/α is given below.

β/α	upper bound on \mathfrak{A}	upper bound on \mathfrak{B}
1.0	1.0000	1.0000
1.5	1.0417	1.0171
2.0	1.1250	1.0481
2.5	1.2250	1.0816
3.0	1.3333	1.1146

REFERENCES

1. P. G. Laurson and W. J. Cox, *Mechanics of materials*, Wiley, 1938, pp. 345-349.
2. W. H. McAdams, *Heat transmission*, McGraw-Hill, 1954, pp. 11-19.
3. G. Pólya and G. Szegő, *Aufgaben und Lehrsätze der Analysis*, vol. I, Dover, 1945, pp. 57 and 214.

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