A LATTICE CHARACTERIZATION OF COMPLETELY REGULAR G_δ-SPACES[1]

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Introduction. In a study of the family of continuous real-valued functions defined on a topological space, Chittenden [4][2] introduced a class of topological spaces characterized by the property that each point of the space is a zero-set for some continuous function. In this paper we consider a variant of this concept which is embodied in the following

DEFINITION 1. A topological space is a G_δ-*space* in case each point of the space is a G_δ.

In the first section we investigate a few of the properties of G_δ-spaces, and show, in particular, that for completely regular spaces Definition 1 agrees with that of Chittenden. Burrill [3] has proved that completely regular G_δ-spaces are characterized by their lattice of zero-sets. Our main result is that the lattice of all real-valued continuous functions on a completely regular G_δ-space characterizes the space. We devote sections two and three to the proof of this result. This proof is obtained by means of techniques similar to those of Shirota [10].

The author wishes to acknowledge his gratitude to Professor M. F. Smiley for his kind and patient advice during the preparation of this paper.

1. G_δ-spaces. In this first section our aim is to investigate the separation and covering properties of the class of G_δ-spaces, to study the Cartesian products of G_δ-spaces, and to obtain a characterization of G_δ points of completely regular spaces.

Obviously, topological spaces which satisfy the first axiom of countability at each point and, in particular, spaces which are metrizable are G_δ-spaces. Also, the perfectly normal spaces are G_δ-spaces. As subsequent examples will show, the class of G_δ-spaces is far broader than the class of spaces satisfying the first axiom of countability. We observe that every G_δ-space is T_1; but, as the following example shows, G_δ-spaces need not be Hausdorff.

Presented to the Society, September 1, 1954; received by the editors August 6, 1954 and, in revised form, December 21, 1954.

[1] This paper is a portion of a thesis submitted by the author to the Graduate College of the State University of Iowa in partial fulfillment of the requirements for the degree of Doctor of Philosophy in August, 1954.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

EXAMPLE 1. Let X be the natural numbers topologized by defining as an open base all subsets of the form $F\cup[k;\ k\geqq m]$ where F is a finite subset of X and m is a natural number. Then, as is easily verified, X is a G_δ-space which fails to be Hausdorff. We note that X is both countable and connected, and, therefore, the only real valued continuous functions on X are constants. Thus, the points of X are not zero-sets.

Next, we give an example due to Hewitt [6] of a nonregular Hausdorff G_δ-space.

EXAMPLE 2. Let X be the set of all real numbers topologized by taking as an open base the usual open intervals with at most a countable set of points deleted. Then X is clearly a Hausdorff G_δ-space, but it fails to be regular.

Bing [1, Example B] has given an example of a non-normal, locally metrizable space. Hence, regular G_δ-spaces need not be normal. Our next example shows that there exist completely regular G_δ-spaces which fail to satisfy the first axiom of countability.

EXAMPLE 3. Let X be the polar plane $[(r,\theta);\ r\geqq 0,\ 0\leqq\theta<2\pi]$. For $(r,\theta)\in X$, $r>0$, define a basis of neighborhoods to be all sets of the form $[(r_1,\theta);\ |r_1-r|<\epsilon]$ where $0<\epsilon<r$. Define as a basis of neighborhoods of $(0, 0)$ the sets of all points (r,θ), $0\leqq r<\epsilon$ and $0\leqq\theta<2\pi$, except for at most finitely many values of θ. Then it is easy to establish that X is a G_δ-space, but the first axiom of countability fails at $(0, 0)$. In addition X is obviously completely regular.

We note that the space of Example 3 is not locally compact. In fact, as was pointed out by Chittenden [4, p. 317], a locally compact Hausdorff G_δ-space satisfies the first axiom of countability at each of its points.

For the Cartesian product of separable metrizable spaces to be metrizable, it is both necessary and sufficient that the number of factors be countable. Similarly, for G_δ-spaces we have

THEOREM 1. *A necessary and sufficient condition that the Cartesian product of a family of G_δ-spaces, each having at least two points, be a G_δ-space is that the family be countable.*

PROOF. The sufficiency is obvious. The necessity is an immediate consequence of the fact that if the number of factors was uncountable, then any countable family of open sets in the product space would have at least one common component consisting of the entire factor space.

Now, let X be a completely regular topological space and denote by $C_p(X)$ the space of all real-valued continuous functions defined on X, endowed with the topology of simple convergence on finite subsets of X. That is, as a sub-basis of open subsets for $C_p(X)$ take all subsets of the form $[f\in C_p(X); |f(x)-f_0(x)| < \epsilon]$ for $x\in X, f_0\in C_p(X)$, and $\epsilon>0$. This is the p-topology in [7]. It is well known that $C_p(X)$ is regular. If A is a dense subset of X and R_x is, for each $x\in A$, homeomorphic to the real numbers in their usual topology, then it is clear that $C_p(X)$ is homeomorphic to a dense subset of the Cartesian product space $\Pi(R_x; x\in A)$ with the property that each projection of this subset onto each factor consists of the entire factor. Recalling that a separable space is one with a countable dense subset, we prove

COROLLARY. *The following statements are equivalent*:
(a) *X is separable,*
(b) *$C_p(X)$ is a separable metric space,*
(c) *$C_p(X)$ has at least one G_δ point.*

PROOF. (a)$\rightarrow$(b). Since X is separable, there is a countable dense subset A of X. Then $\Pi(R_x; x\in A)$ is a separable metric space; hence, $C_p(X)$, as a dense subset, is a separable metric space.

(b)$\rightarrow$(c). Every metrizable space is a G_δ-space.

(c)$\rightarrow$(a). If X is nonseparable, then, by the above remarks and Theorem 1, $C_p(X)$ is not a G_δ-space. Hence, there is an $f\in C_p(X)$ which is not a G_δ. However, the neighborhoods of any $g\in C_p(X)$ are obtained from those of f by translation. Therefore, no point of $C_p(X)$ is a G_δ if X is nonseparable.

We observe that $C_p(X)$ is a topological group under the usual operation of addition. It is known that a topological group is metrizable if and only if it satisfies the first axiom of countability. As a result, we are led to ask whether there exist any topological groups which are G_δ-spaces but are not metrizable. Certain of the $\mathcal{LF}$-spaces of Dieudonné and Schwartz [5] provide an affirmative answer.

Our final result of this section is a very useful characterization in terms of continuous functions of G_δ points in completely regular spaces.

THEOREM 2. *An element x of a completely regular space X is a G_δ if and only if there is a function $f\in C(X)$ such that $f(x)\neq f(y)$ whenever $x\neq y$.*

The proof is quite easy and will therefore be omitted.

The class of spaces considered by Chittenden consists of those spaces which satisfy the condition of Theorem 2. It is not difficult to see that this condition is satisfied by spaces more general than completely regular G_δ-spaces; however, in the following two sections, little, if anything, is to be gained by considering this slight generalization.

2. **The lattice $C_+(X)$.** The family of all real-valued continuous functions defined on a topological space X forms a distributive lattice [2], which we denote by $C(X)$, where $(f\vee g)(x)=\max(f(x), g(x))$ and $(f\wedge g)(x)=\min(f(x), g(x))$ for all $x\in X$. In this section we review some of the properties of $C_+(X)$, the sublattice of $C(X)$ consisting of all non-negative functions. We denote the least element of $C_+(X)$ by 0.

Throughout the remainder of the paper X will stand for a completely regular space. Furthermore, if A is a subset of a topological space, then A^- will denote the closure of A and A' will denote the complement of A. Thus, $A^{-\prime-\prime}$ is the interior of A^-.

DEFINITION 2 (Shirota [10]). We define the binary relations $\subset$ and $\ll$ on $C_+(X)$ by

(i) $f\subset g$ in case $g\wedge h=0$ implies $f\wedge h=0$, for all $h\in C_+(X)$;

(ii) $f\ll g$ in case every subset $[h_\alpha;\ \alpha\in\Omega]$ of $C_+(X)$, which has an upper bound in $C_+(X)$ and is such that $h_\alpha\subset f(\alpha\in\Omega)$, has an upper bound $h\subset g$.

We note that both of the relations $\subset$ and $\ll$ are transitive. If $f\in C_+(X)$, then we set $P(f)=[x\in X;\ f(x)>0]$.

The following lemma is due to Pierce [9].

LEMMA 2.1. *Let f, $g\in C_+(X)$. Then*

(a) $P(f\wedge g)=P(f)\cap P(g)$,

(b) $P(f\vee g)=P(f)\cup P(g)$,

(c) $P(f\wedge g)^{-\prime-\prime}=P(f)^{-\prime-\prime}\cap P(g)^{-\prime-\prime}$,

(d) $f\subset g$ *if and only if* $P(f)^-\subset P(g)^-$.

We say that two subsets A and B of X are *completely separated* in X in case there is an $f\in C_+(X)$ such that $f(x)=0$ $(x\in A)$ and $f(x)=1$ $(x\in B)$. The following lemma is due to Shirota [10].

LEMMA 2.2. (a) *$f\ll g$ if and only if $P(f)^-$ and $P(g)^{-\prime-}$ are completely separated in X,*

(b) *$f_1\ll g_1$ and $f_2\ll g_2$ imply $f_1\wedge f_2\ll g_1\wedge g_2$.*

Next, we note for future use the following obvious facts about the usual ring operations of positive continuous functions.

LEMMA 2.3. *If f and g are positive real-valued continuous functions on X, then $P(fg)=P(f\wedge g)$ and $P(f+g)=P(f\vee g)$.*

LEMMA 2.4. *Let $x\in P(g)^{-\prime-\prime}$ for some $g\in C_+(X)$. Then there exists an*

$f\in C_+(X)$ such that $f\ll g$ and $x\in P(f)$.

PROOF. Let U be an open set such that $x\in U\subset U^-\subset P(g)^{-\prime-\prime}$ and $\phi\in C_+(X)$ such that $\phi(x)=2$ and $\phi(y)=0$ $(y\notin U)$; thus, $\phi\subset g$. Set $V=[z;\ \phi(z)>1]$ and choose $f\in C_+(X)$ such that $f(x)=1$ and $f(z)=0$ $(z\notin V)$. Then $\phi(z)\geqq 1$ $(z\in P(f)^-)$ and $\phi(y)=0$ $(y\in P(g)^{-\prime-})$, since $U\subset P(g)^{-\prime-\prime}$ implies $U\cap P(g)^{-\prime-}=0$. Hence, $P(g)^{-\prime-}\subset U'\subset P(\phi)'$, so that $x\in P(f)$ and $f\ll g$.

Our final lemma, due to Kaplansky [**8**], states, in effect, that $C_+(X)$ is completely determined by $C(X)$.

LEMMA 2.5. *If $f\in C(X)$ and $C_f(X)=[g\in C(X);\ g\geqq f]$, then $C_f(X)$ is lattice isomorphic to $C_+(X)$.*

3. **The characterization theorem.** It is evident from Lemmas 2.1 and 2.2 that the relations $\subset$ and $\ll$ on the lattice $C_+(X)$ yield a considerable amount of information about the space X. In fact, Pierce [**9**] has shown by means of the former relation that the lattice of regular open sets of X is characterized by $C_+(X)$. In this section we establish, by means of both relations, that completely regular G_δ-spaces are characterized by the lattice $C(X)$. Throughout this section X will denote a completely regular G_δ-space, although Lemmas 3.1, 3.2, 3.3 hold in any completely regular space.

DEFINITION 3. A countable subset $\{f_n\}$ of $C_+(X)$ is called a *G-set* in case

(1) $0<f_{n+1}\ll f_n$ for $n=1, 2, \cdots$,

(2) $\bigwedge_{n=1}^{\infty} f_n=0$.

We call a G-set $\{f_n\}$ *free* in case whenever $\{h_n\}\subset C_+(X)$ is such that $h_n\subset f_n\subset h_n$ $(n=1, 2, \cdots)$, then $\{h_n\}$ has an upper bound in $C_+(X)$. A G-set which is not free will be called *fixed.* Let $\mathcal{G}$ denote the family of all fixed G-sets of $C_+(X)$. A set $\{f_n\}$ in $\mathcal{G}$ is *irreducible* provided whenever $\{g_n\}$, $\{h_n\}$ in $\mathcal{G}$ are such that $\{g_n\wedge f_n\}$ and $\{h_n\wedge f_n\}$ are in $\mathcal{G}$, then $\{g_n\wedge h_n\}$ is in $\mathcal{G}$. Let $\mathfrak{I}$ denote the family of all irreducible fixed G-sets of $C_+(X)$.

We now pause briefly to outline the proof of the desired characterization theorem. We first show that a G-set $\{f_n\}$ is fixed if and only if $\bigcap_{n=1}^{\infty} P(f_n)^-\neq 0$, and that it is irreducible if and only if this intersection consists of a single point. Next we establish that each element of X is in the intersection $\bigcap_{n=1}^{\infty} P(f_n)^-$ for some irreducible fixed G-set $\{f_n\}$. Then an equivalence relation is defined on $\mathfrak{I}$ in an obvious way, and the equivalence classes are endowed with a variation of the Stone topology. The resulting topological space is shown to be homeomorphic to X, from which we conclude that $C_+(X)$ characterizes X as a completely regular G_δ-space. The final result is then obtained by an application of Lemma 2.5.

LEMMA 3.1. *If* $\{f_n\}$ *and* $\{g_n\}$ *are G-sets, then*

$$\bigcap_{n=1}^{\infty} P(f_n \wedge g_n)^- = \left[\bigcap_{n=1}^{\infty} P(f_n)^-\right] \cap \left[\bigcap_{n=1}^{\infty} P(g_n)^-\right].$$

PROOF. By Lemmas 2.1(c) and 2.2(a) and (b), $P(f_{n+1})^- \cap P(g_{n+1})^- \subset P(f_n)^{-\prime-\prime} \cap P(g_n)^{-\prime-\prime} = P(f_n \wedge g_n)^{-\prime-\prime} \subset P(f_n \wedge g_n)^- \subset P(f_{n-1} \wedge g_{n-1})^{-\prime-\prime} = P(f_{n-1})^{-\prime-\prime} \cap P(g_{n-1})^{-\prime-\prime} \subset P(f_{n-1})^- \cap P(g_{n-1})^-$. Hence, $[\bigcap_{n=1}^{\infty} P(f_n)^-] \cap [\bigcap_{n=1}^{\infty} P(g_n)^-] \subset \bigcap_{n=1}^{\infty} P(f_n \wedge g_n)^- \subset [\bigcap_{n=1}^{\infty} P(f_n)^-] \cap [\bigcap_{n=1}^{\infty} P(g_n)^-]$.

LEMMA 3.2. *If* $\{f_n\}$ *is in* $\mathcal{G}$, *then* $\bigcap_{n=1}^{\infty} P(f_n)^- \neq 0$.

PROOF. Suppose $\bigcap_{n=1}^{\infty} P(f_n)^- = 0$ and $h_n \subset f_n \subset h_n$ $(n=1, 2, \cdots)$. Then, by Lemma 2.1(d), $P(h_n)^- = P(f_n)^-$ $(n=1, 2, \cdots)$. If $\{h_n\}$ can be shown to possess an upper bound in $C_+(X)$, then it follows that $\{f_n\}$ is free. Define

$$h(x) = \sum_{n=1}^{\infty} h_n(x) \qquad (x \in X).$$

Now $m > n$ implies $f_m \ll f_n$, so that by Lemma 2.2(a), $P(h_m)^- = P(f_m)^- \subset P(f_n)^{-\prime-\prime} \subset P(f_n)^- = P(h_n)^-$. Hence, $h(x) = \sum_{i=1}^{n} h_i(x)$ for all $x \in P(f_n)^{-\prime}$. Thus, h is continuous at every $x \in \bigcup_{n=1}^{\infty} P(f_n)^{-\prime} = X$. Therefore, since $h_n \leq h$ $(n=1, 2, \cdots)$ and $h \in C_+(X)$, $\{f_n\}$ is free.

LEMMA 3.3. *If* $\{f_n\}$ *is a G-set and* $\bigcap_{n=1}^{\infty} P(f_n)^- \neq 0$, *then* $\{f_n\}$ *is fixed.*

PROOF. Let $x \in \bigcap_{n=1}^{\infty} P(f_n)^-$. Since $f_{n+1} \ll f_n$ $(n=1, 2, \cdots)$, it follows from Lemma 2.2(a) that $x \in \bigcap_{n=1}^{\infty} P(f_n)^{-\prime-\prime}$. Thus, since X is completely regular, for each positive integer n there exists a function $g_n \in C_+(X)$ such that $g_n(x) = n$ and $g_n(y) = 0$ for all $y \in P(f_n)^{-\prime-}$. Then $P(g_n) \subset P(f_n)^{-\prime-\prime-} = P(f_n)^-$. Let $h_n = g_n + f_n$ $(n=1, 2, \cdots)$. Then, by Lemma 2.3, $P(h_n)^- = P(g_n)^- \cup P(f_n)^- = P(f_n)^-$, so that $h_n \subset f_n \subset h_n$ $(n=1, 2, \cdots)$. However, since $h_n(x) \geq n$ for each n, $\{h_n\}$ does not have an upper bound in $C_+(X)$, and therefore, $\{f_n\}$ is fixed.

LEMMA 3.4. *Let* $x \in X$ *and* $\{U_{x,n}\}$ *be a countable family of neighborhoods of* x *such that* $U_{x,m}^- \subset U_{x,n}$ $(m > n)$ *and* $[x] = \bigcap_{n=1}^{\infty} U_{x,n}$. *Let* $f_{x,n} \in C_+(X)$ $(n=1, 2, \cdots)$ *be such that* $f_{x,n}(x) = 1/n$, $f_{x,n}(y) = 0$ $(y \notin U_{x,n})$, *and* $f_{x,m} \ll f_{x,n}$ *for* $m > n$. *Then* $\{f_{x,n}\}$ *is in* $\mathfrak{J}$.

PROOF. We observe first that since X is completely regular and x is a G_δ, the sequence $\{U_{x,n}\}$ exists in X, and, by Lemma 2.4, the functions $f_{x,n}$ $(n=1, 2, \cdots)$ exist in $C_+(X)$. Also, it is clear that

$\{f_{x,n}\}$ is a G-set, and, since $\{n^2 f_{x,n}\}$ fails to have an upper bound in $C_+(X)$, it follows that $\{f_{x,n}\}$ is fixed. Now let $\{g_n\}$, $\{h_n\}$ in $\mathcal{G}$ be such that $\{g_n \wedge f_{x,n}\}$ and $\{h_n \wedge f_{x,n}\}$ are in $\mathcal{G}$. Then, by Lemma 3.2, $\bigcap_{n=1}^\infty P(g_n \wedge f_{x,n})^- \neq 0$, and, since $\bigcap_{n=1}^\infty P(g_n \wedge f_{x,n})^- \subset \bigcap_{n=1}^\infty P(f_{x,n})^- = [x]$, it follows that $[x] = \bigcap_{n=1}^\infty P(g_n \wedge f_{x,n})^-$. Thus, similarly, $[x] = \bigcap_{n=1}^\infty P(h_n \wedge f_{x,n})^-$. By Lemma 2.2(b), $g_m \wedge f_{x,m} \ll g_n \wedge f_{x,n}$ for $m > n$, and consequently, $P(g_m \wedge f_{x,m})^- \subset P(g_n \wedge f_{x,n})^{-\prime-\prime} \subset P(g_n)^{-\prime-\prime}$ for $m > n$. Thus, $x \in P(g_n)^-$ $(n = 1, 2, \cdots)$, and similarly, $x \in P(h_n)^-$ $(n = 1, 2, \cdots)$. Hence, by Lemma 3.1, $x \in \bigcap_{n=1}^\infty P(g_n \wedge h_n)^-$ and also $g_n \wedge h_n > 0$ $(n = 1, 2, \cdots)$. Then from Lemma 2.2(b) and the fact that $\bigwedge_{n=1}^\infty (g_n \wedge h_n) \leqq \bigwedge_{n=1}^\infty g_n = 0$, it follows that $\{g_n \wedge h_n\}$ is a G-set. However, since $x \in \bigcap_{n=1}^\infty P(g_n \wedge h_n)^-$, Lemma 3.3 implies that $\{g_n \wedge h_n\}$ is in $\mathcal{G}$, and thus $\{f_{x,n}\}$ is in $\mathfrak{J}$.

LEMMA 3.5. *If $\{f_n\}$ is in $\mathfrak{J}$, then $\bigcap_{n=1}^\infty P(f_n)^-$ is a single point of X.*

PROOF. By Lemma 3.2, $\bigcap_{n=1}^\infty P(f_n)^- \neq 0$. Suppose $x, y \in \bigcap_{n=1}^\infty P(f_n)^-$ with $x \neq y$. Since X is completely regular, there exist open sets U and V such that $x \in U$, $y \in V$, and $U^- \cap V^- = 0$. Now we may choose $\{f_{x,n}\}$ and $\{f_{y,n}\}$ as in Lemma 3.4 such that $P(f_{x,n}) \subset U$ and $P(f_{y,n}) \subset V$ for $n = 1, 2, \cdots$. Then clearly $\{f_{x,n} \wedge f_{y,n}\}$ is not a G-set. Since $x \in \bigcap_{n=1}^\infty P(f_n)^-$ and $x \in \bigcap_{n=1}^\infty P(f_{x,n})$, it follows that $P(f_{x,n}) \cap P(f_n) = P(f_{x,n} \wedge f_n) \neq 0$; thus, $f_{x,n} \wedge f_n > 0$ $(n = 1, 2, \cdots)$. From Lemma 2.2(b) and the fact that $f_{x,n} \wedge f_n \leqq f_n$ we conclude that $\{f_{x,n} \wedge f_n\}$ is a G-set. Moreover, by Lemma 3.1, $x \in [\bigcap_{n=1}^\infty P(f_{x,n})^-] \cap [\bigcap_{n=1}^\infty P(f_n)^-]$ implies $x \in \bigcap_{n=1}^\infty P(f_{x,n} \wedge f_n)^-$. Therefore, $\{f_{x,n} \wedge f_n\}$ is in $\mathcal{G}$, and, similarly, $\{f_{y,n} \wedge f_n\}$ is in $\mathcal{G}$. This contradicts the hypothesis that $\{f_n\}$ is in $\mathfrak{J}$. Thus, $\bigcap_{n=1}^\infty P(f_n)^-$ is a single point of X.

DEFINITION 4. We define the binary relation $\sim$ on $\mathfrak{J}$ by $\{f_n\} \sim \{g_n\}$ in case $\{f_n \wedge g_n\}$ is in $\mathfrak{J}$.

LEMMA 3.6. *$\{f_n\} \sim \{g_n\}$ if and only if $\bigcap_{n=1}^\infty P(f_n)^- = \bigcap_{n=1}^\infty P(g_n)^-$.*

PROOF. By Lemma 3.5, there are unique elements $x, y \in X$ such that $[x] = \bigcap_{n=1}^\infty P(f_n)^-$ and $[y] = \bigcap_{n=1}^\infty P(g_n)^-$. Also, by Lemma 3.5, it follows that $\{f_n \wedge g_n\}$ is in $\mathfrak{J}$ if and only if there is an unique element $z \in X$ such that $[z] = \bigcap_{n=1}^\infty P(f_n \wedge g_n)^-$. Hence, by Lemma 3.1, $\{f_n\} \sim \{g_n\}$ if and only if $x = y = z$.

The following important lemma is now an immediate consequence of Lemma 3.6.

LEMMA 3.7. *The relation $\sim$ is an equivalence relation.*

DEFINITION 5. Let $\mathfrak{J}^\sim$ be the family of all equivalence classes $\{f_n\}^\sim$ of $\mathfrak{J}$ determined by $\sim$. Let $\mathfrak{B}$ be the set of all subsets of $\mathfrak{J}^\sim$

of the form

$$B(f) = [\{f_n\}^\sim; f_m \ll f \text{ for some } f_m]$$

for all $f \in C_+(X)$.

LEMMA 3.8. *$\mathcal{B}$ is an open basis of a topology for $\mathfrak{J}^\sim$. We denote the resulting topological space by $S(\mathfrak{J}^\sim, X)$.*

PROOF. It will suffice to show that $B(f) \cap B(g) = B(f \wedge g)$. Clearly, $B(f \wedge g) \subset B(f) \cap B(g)$. Let $\{f_n\}^\sim \in B(f) \cap B(g)$. Then there are $\{f_n\}$ and $\{f_n^*\}$ in $\{f_n\}^\sim$ such that $f_k \ll f$ and $f_p^* \ll g$ for some integers k and p. By Lemma 2.2(b), $f_m \wedge f_m^* \ll f \wedge g$ where $m \geqq k, p$. Also, by Lemma 3.7, $\{f_n \wedge f_n^*\}$ is in $\{f_n\}^\sim$. Hence, $\{f_n\}^\sim$ is in $B(f \wedge g)$, and thus $\mathcal{B}$ forms an open basis for $\mathfrak{J}^\sim$.

LEMMA 3.9. *The space $S(\mathfrak{J}^\sim, X)$ is homeomorphic to X under the mapping $x \rightarrow \{f_{x,n}\}^\sim$.*

PROOF. By Lemmas 3.4, 3.5, and 3.6, the mapping is one to one of X onto $\mathfrak{J}^\sim$. We now note that since X is completely regular, the sets $P(f)^{-\prime-\prime}$ ($f \in C_+(X)$) form a basis of open sets for X. If $x \in P(f)^{-\prime-\prime}$, then by Lemma 3.4, there is an irreducible G-set $\{f_{x,n}\}$ such that $P(f_{x,1})^- \subset P(f)^{-\prime-\prime}$ and thus, by Lemma 2.2(a), $f_{x,2} \ll f$, so that $\{f_{x,n}\}^\sim \in B(f)$. On the other hand, if $\{f_{x,n}\}^\sim \in B(f)$, then $f_{x,m} \ll f$ for some m. However, by Lemma 2.2(a), $P(f_{x,m})^- \subset P(f)^{-\prime-\prime}$; thus, $x \in P(f)^{-\prime-\prime}$. Consequently, the mapping is bicontinuous.

Now we are able to obtain the desired characterization.

THEOREM 3. *A completely regular G_δ-space is characterized by its lattice of real-valued continuous functions.*

PROOF. Let X and Y be completely regular G_δ-spaces such that $C(X)$ and $C(Y)$ are isomorphic. Then, by Lemma 2.5, $C_+(X)$ and $C_+(Y)$ are isomorphic. Thus, $S(\mathfrak{J}^\sim, X)$ and $S(\mathfrak{J}^\sim, Y)$ are homeomorphic, and finally, by Lemma 3.8, X and Y are homeomorphic.

Shirota [10] proved that the lattice $C(X)$ also characterizes spaces which are locally compact and paracompact or Q-spaces in the sense of Hewitt [7]. Example 3 shows that our result is not subsumed in the first of these characterizations and the space T_Ω of all ordinals less than the first uncountable ordinal [7; p. 63], which is a completely regular non-Q-space satisfying the first axiom of countability, shows that Theorem 3 is not subsumed in the Q-space characterization.

The ring of real-valued continuous functions defined on a space characterizes the lattice (see [8]); thus we obtain as a corollary to Theorem 3 the result that completely regular G_δ-spaces are characterized by their rings of real-valued continuous functions. This latter result has been obtained independently by L. E. Pursell.

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